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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN OPTIMAL SOLUTION TO A CONTROL PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

Abstract. This work examines an optimal control problem for a class of integro-differential equation parabolic type in a Banach space. Sufficient conditions on the coefficients are established to guaranty the existence of optimal pairs. In our case, the main problem of optimal control is considered up to the time of existence of the solution from the fixed domain, which distinguishes the analyzed problem from other investigations.

Keywords: optimal control, weakly convergence, unbounded operator, semigroup, minimizer, fixed point.

We consider an optimal control for an evolutionary integro-differential equation in Banach spaces involving the time until the solution leaves a specified domain. The system is governed by the following:

$$\frac{du}{dt} = Au + f_1(t,u) + f_2(t,u)z + \int_0^t f_3(t,s,u(s))z(s)ds\,,$$

$$u(0) = u_0. \qquad (1)$$

The representation of the control is evaluated using a criterion:

$$J\,[z] = \int_0^\tau L(t,u(t),z(t))dt \to \inf\,. \qquad (2)$$

Here $A : X \to X$ is a linear (unbounded) operator in the reflexive Banach space X, where X is equipped with the norm $\|\cdot\|_X$, domain $D(A)$ is dense in X.

Throughout this paper, we will use $\|\cdot\|$ instead of $\|\cdot\|_X$. Let D be an open set in X, and let ∂D be its boundary in the sense of [1, p. 8], and $\overline{D} = D \cup \partial D, \tau$ be the first moment of the solution $u(t)$ exiting at ∂D until T, otherwise $\tau = T, T > 0$ is fixed. Let

$f_1\colon [0,T] \times D \to X, f_2\colon [0,T] \times D \to L(X,X)$ be the space of linear bounded operators from X to X with the norm $\|\cdot\|_{\mathcal{L}}, f_3\colon [0,T] \times [0,T] \times D \to L(X,X), z(t) \in X$ be the control parameter.

Our work summarizes the results of [2], [3], [4] and [6] for the finite-dimensional case.

Assumption 1. *(H1). We assume that operator A is the infinitesimal generator of a C_0 semigroup $\{S(t)_t \geq 0\}$ bounded linear operators in X. We assumed that the semigroup $S(t)$ is compact for $t > 0$.*

Then, under assumption (H1), theorem 3.2 of [5] implies that the semigroup $S(t)$ is continuous in the uniform operator topology for $t > 0$:

Definition 1. We say that the function $u(t) \in X$ is a mild solution of the initial problem

$$\frac{du}{dt} = \dot{u} = Au + f\left(t, u(t), \int_0^t \varphi\left(t, s, x(s)\right) ds\right), u(0) = u_0 \in X. \tag{3}$$

on $[0,T]$ if:

1) $u(0) = u_0$;

2) $u \in C([0,T], X)$;

3) $u(t)$ satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\, f\left(s, u(s), \int_0^s \varphi\left(s, \tau, u(\tau)\right) d\tau\right) ds, \tag{4}$$

on $[0,T]$.

In the following, we will interpret the solution to Problem (1) in the sense of definition 1.

We consider the optimal control problem (1)-(2) under the following conditions.

Assumption 2. *(H2). The admissible control $z(t)$ are the functions $z(t)$ such that $z \in L^P((0,T); X), p > s$, and $z(t) \in F$ for a.e. $t \in (0,T)$, where F is closed convex set in X, and the solution of (1). $u(t) = u(t,z)$, that corresponds control z exists and continuous up to the first moment it reaches the boundary of the domain D, and $J[z] < \infty$.*

We impose the following conditions of nonlinearity:

Assumption 3. *(H3). Let* $f_1 : [0,T] \times D \to X, f_2: [0,T] \times D \to L(X,X), f_3: [0,T] \times [0,T] \times D \to L(X,X)$, *and for any* $t \in [0,T], s \in [0,T], f_1(t,\cdot)$, $f_2(t,\cdot), f_3(t,s,\cdot)$ *are continuous on* $u \in D$, *and for any fixed* $u \in D, f_1(\cdot,u)$, $f_2(\cdot,u), f_3(\cdot,\cdot,u)$ *are measurable on* t.

Assumption 4. *(H4). Linear growth. There exist positive constant* K, *such that*

$$\| f_1(t,u) \| + \| f_2(t,u) \|_{\mathcal{L}} + \| f_3(t,s,u) \|_{\mathcal{L}} \leq K(1+|u|)$$

for all $t \in [0,T], u \in D$.

Now we will describe conditions of the cost function.

Assumption 5. *(H5). The mapping* $L(t,u,z): [0,T] \times D \times F \to \mathbb{R}^1$ *in continuous for all variables.*

Assumption 6. *(H6). The Fréchet derivative* L_z *is continuous for all argument and there exist* $C_1 > 0, \alpha > 0$ *such that*

$$\| L_z(t,u,z) \|_* \leq C_1(1 + \| u \|^{\alpha} + \| z \|^{P-1}),$$

where $\|\cdot\|_*$ is the norm in the dual space X^*.

Assumption 7. *(H7). There exist constants* $C_2 \geq 0, C_3 > 0$, *and function* $\omega \in L^1(0,T;R^+)$ *such that*

$$L(t,u,z) \geq \omega(t) + C_2 \| u \| + C_3 \| z \|^P ,$$

for all $t \in [0,T], u \in D, z \in F$.

Assumption 8. *(H8).* $L(t,u,z)$ *is convex on z for fixed* $t \in [0,T], u \in D$.

Assumption 9. *(H9). There exist* $C_4 > 0, \upsilon > 0$ *such that for any* $u, u_1 \in X$, $t \in [0,T], z \in P : |L(t,u,z)) - L(t,u_1,z)| \leq C_0 \| u - u_1 \| (1 + \| u \|^{\upsilon} + \| u_1 \|^{\upsilon} + \| z \|^P)$.

The main result of this work concerns the existence of an optimal pair $(u^*(t), z^*(t))$ for the problem (1) - (2).

The main result is stated in the theorem:

Theorem 1. *Suppose that assumptions (H1) - (H9) hold. Then the optimal control problem (1) - (2) has a solution* $(u^*(t), z^*(t))$*, where* $z^*(t)$ *is an optimal control,* $u^*(t)$ *is the corresponding optimal trajectory.*

Remark 1. *It is clear that the result of theorem 1 is valid if we assume that the mapping* $z(t)$ *takes values in another reflexive Banach space* Y*, with obvious minor modifications to the conditions. The belonging* $z(t) \in X$ *is assumed solely for simplicity.*

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