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MATHEMATICAL MODEL OF COMPLEX MOVEMENT OF A MATERIAL POINT ON A SURFACE OF AGRICULTURAL MACHINE WORKING BODY

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Abstract. *In theoretical study of most of agricultural machines working bodies there is a need of the modeling of material particles (material points) motion on their working surfaces. Questions of such modeling in cases when the specified surfaces are give to material points movement with difficult trajectories are especially difficult. Objective of this research is to develop the basic provisions of complex movement theory of material point. When carrying out research methods of modeling theory, theoretical mechanics, higher mathematics, in particular differential geometry, methods of drawing up programs and numerical calculations on the personal computer are used. As a result of the conducted theoretical research the complex movement of material point which relative movement happens in a moving trihedron of curve which is defined by the natural equations is considered. The figurative movement of a trihedron is defined by differential characteristics of curve. Competency of use of Frenet formulas for finding absolute speed and point acceleration in projections on unit vectors of the moving trihedron is proved. As a result of numerical calculations on the personal computer there were found absolute trajectories of material point movement and qualitative assessment of received results was carried out.*
KEYWORDS: *TRIHEDRON, COMPLEX MOVEMENT, MATERIAL POINT, ABSOLUTE SPEED, ACCELERATION, THEORY*

1. Introduction

The motion of the material point along the plane (the gravitational surface, the rough plane, etc.) has been the subject of research of many scientists, with a worldwide reputation ranging from Galileo, Huygens, Newton, Euler, Ostrogradsky and others.As a most fundamental studies of the motion of a material particle on frictional surfaces of agricultural machines should be considered the works of academician Vasilenko P.M [1] and other domestic scientists academicians Zaika P. M, Berg B. A, and also Grigorieva S. M., Melnikov S. V. and others. A considerable number of analytical problems in the theory of agricultural machines still need the application of the theory of motion of a material point (particle) or a solid body over surfaces that are used in the design of new structures.

2. Preconditions and means for resolving the problem

2.1. Formulation of the problem

The theory of the complex motion of a material point has a completed form and does not even need any refinement. It is based on the fact that the motion of a point is investigated simultaneously with respect to two coordinate systems. One of them (the main one) is assumed to be fixed, and the second one is providing relative motion according to a given law in relation to the fixed point. Generally, the relative motion of the material point is carried out in the relative movement of the coordinate system. The sum of these movements (relative and portable) creates the absolute motion of the material point with respect to the basic coordinate system. In this case, the movements (both portable and relative) are usually given by the dependencies in the time function.

There is also known the natural way of specifying the motion of a material point, in which the velocity and acceleration are considered in the projections onto the units of the accompanying trihedron of the trajectory (the Frenet trihedron). However, in the available literature it is not possible to find the application of the Frenet trihedron as a moving coordinate system in which the material point is moving relatively. The development of the theory of the complex motion of a material point along the horizontal plane with the use of the Frenet trihedron is the subject of our investigation

2.2. Analysis of recent research and publications

The natural way of specifying the motion of a material point is considered quite well known and is widely used in studies on many issues in the field of mechanization of agriculture and the theory of agricultural machines. In this case, the vast majority of simple motion of material points is considered. There are known the examples with the use of a trihedron and Frenet formulas when considering the motion of a rigid body in its system, for example, an aircraft [2]. The kinematics of the motion of the accompanying trihedron of a helical line is considered in [3]. In the latest scientific and educational publications, the kinematics of the accompanying trihedron of a trajectory as a rigid body are either not considered at all, or are considered with reference to earlier studies and publications [4, 7, 8]. Meanwhile, as shown in [5, 9, 10, 11], the trihedron and Frenet's formulas can be successfully used in problems of kinematics and the dynamics of the complex motion of a material point, in particular when considering issues that are related to the study of agricultural machines.

2.3. Purpose of the study

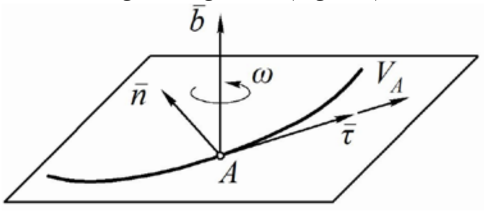
As a main aim of our study is the further development of the theory of the complex motion of a material point along the plane with the application of the accompanying trihedron of the curve and the Frenet formulas.

3. Results and Discussion

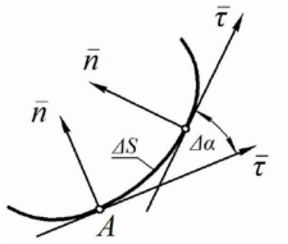
At any point of the curve, three mutually perpendicular directions can be constructed. Single units along them (tangent, principal normal and binormal) form the accompanying (natural) trihedron of the curve or the triaxial Frenet. For a planar curve, the unit vectors u are in the plane of the curve, and the unit vector is perpendicular to it (see Fig. 1, a).

If you move the trihedron at a given speed V_A along the curve, you can determine the velocity and acceleration of any point of the trihedron, the magnitude and direction of which will depend on the curvature of the curve. The velocity of point of the trihedron will consist of velocity of pole (the origin of coordinates *A*) and velocity of this point in the rotational motion of trihedra around the instantaneous rotation axis, which coincides with the binormal hort

$\bar{b}$. During a certain period of time Δt, the trihedron moves along the curve to a new position, due to displacement to a distance Δs and rotation through an angle $\Delta\alpha$ (Fig. 1, b).



a



b

***Fig. 1.** – Accompanying trihedron of the Frenet curve: a) position of the vector of the instantaneous rotation axis; b) to determine the angle of rotation $\Delta\alpha$ when the trihedron is moved along the curve by a distance Δs (the binormal $\bar{b}$ is projected to a point)*

The magnitude of the angular velocity ω can be defined as the limiting ratio of the increment of the angle to the increase in time:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \frac{d\alpha}{dt}. \qquad (1)$$

We pass from the time parameter t to the arc coordinate s (paths along the arc):

$$\omega = \frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = V_A \frac{d\alpha}{ds} = V_A k, \qquad (2)$$

where k – curvature of the curve at the current point A.

Thus, the magnitude of the angular velocity of the trihedron depends on the speed of its motion along the curve and the curvature of the curve itself at the point where the vertex of the trihedron is located.

We fix the point B rigidly in the trihedron system and we find its velocity. The radius-vector $\bar{r}_B$ that determines the position of the point B relative to the fixed coordinate system Oxy (Fig. 2) can be specified with the help of two vectors: $\bar{r}_A$, which determines the position of the vertex of the trihedron in the coordinate system Oxy, and $\bar{\rho}$, which determines the position of point B in the trihedron system. The value of the radius vector $\bar{r}_B$ will be:

$$\bar{r}_B = \bar{r}_A + \bar{\rho}\,. \qquad (3)$$

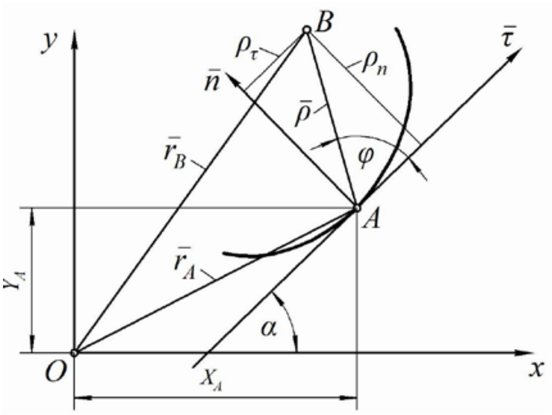


***Fig. 2.** – Position of the point B in two coordinate systems: the immovable Oxy and the movable trihedron of the curve $\bar{\tau}\bar{n}\bar{b}$*

Let the point B in the system of the accompanying trihedron will be given by a vector $\rho = const$ whose components in the projections on unit vectors (orthes) have the value ρ_τ and ρ_n (Fig. 2).

We write the vector sum (3) in the projections on the axis of the fixed coordinate system Oxy. We will have:

$$\begin{aligned} x_B &= x_A + \rho_\tau \cos\alpha - \rho_n \sin\alpha; \\ y_B &= y_A + \rho_\tau \sin\alpha + \rho_n \cos\alpha. \end{aligned} \qquad (4)$$

Differentiating (4) with respect to time t, we find the projection of the velocity of point B on the coordinate axes of the immovable system:

$$\begin{aligned} \frac{dx_B}{dt} &= \frac{dx_B}{ds}\frac{ds}{dt} = V_A \frac{dx_B}{ds} = V_A\left(x'_A - \rho_\tau\alpha'\sin\alpha - \rho_n\alpha'\cos\alpha\right); \\ \frac{dy_B}{dt} &= \frac{dy_B}{ds}\frac{ds}{dt} = V_A \frac{dy_B}{ds} = V_A\left(y'_A + \rho_\tau\alpha'\cos\alpha - \rho_n\alpha'\sin\alpha\right). \end{aligned} \qquad (5)$$

In expressions (5) there was done the transition from the time parameter t to the arc coordinate s – the arc length of the curve. In this case, the components of expressions (5) acquire a geometric content [7]:

$$x'_A = \cos\alpha; \quad y'_A = \sin\alpha; \quad \alpha' = k. \qquad (6)$$

Taking into account (6), the projections of the absolute velocity of the point B in (5) on the axis of the immovable coordinate system are written as follows:

$$\begin{aligned} V_{Bx} &= x'_B = V_A\left[(1 - k\rho_n)\cos\alpha - k\rho_\tau\sin\alpha\right]; \\ V_{By} &= y'_B = V_A\left[(1 - k\rho_n)\sin\alpha + k\rho_\tau\cos\alpha\right]. \end{aligned} \qquad (7)$$

The result (7) can also be obtained from the well-known formula [4]:

$$\overline{V_B} = \overline{V_A} + \bar{\omega} \times \bar{\rho}, \qquad (8)$$

where the first component $\overline{V_A}$ is the velocity of the pole A, and the second $\bar{\omega}\times\bar{\rho}$ is the speed of the point B around the pole. Accordingly they can be found in this way:

$$\begin{aligned} V_{Ax} &= \frac{dx_A}{dt} = \frac{dx_A}{ds}\frac{ds}{dt} = V_A\frac{dx_A}{ds} = V_A x'_A = V_A\cos\alpha; \\ V_{Ay} &= \frac{dy_A}{dt} = \frac{dy_A}{ds}\frac{ds}{dt} = V_A\frac{dy_A}{ds} = V_A y'_A = V_A\sin\alpha. \end{aligned} \qquad (9)$$

$$\bar{\omega}\times\bar{\rho} = \begin{vmatrix} x & y & z \\ 0 & 0 & V_A k \\ \rho_\tau\cos\alpha - \rho_n\sin\alpha & \rho_\tau\sin\alpha + \rho_n\cos\alpha & 0 \end{vmatrix},$$

from where:

$$\bar{\omega}\times\bar{\rho} = \left\{-V_A k\left(\rho_\tau\sin\alpha + \rho_n\cos\alpha\right); V_A k\left(\rho_\tau\cos\alpha - \rho_n\sin\alpha\right)\right\}. \qquad (10)$$

Having added the components of the projections (9) and (10) to the corresponding coordinate axes, we obtain the already known result (7).

And now we show how it is not very difficult to find the absolute velocity of the point B in the projections onto the unit vectors of the accompanying trihedron of the curve. For comparison, we first do this using formula (8), and then applying Frenet formulas. We find the vector $\overline{\omega}\times\overline{\rho}$ in the projections onto the unit vectors of the trihedron:

$$\overline{\omega}\times\overline{\rho} = \begin{vmatrix} \bar{\tau} & \bar{n} & \bar{b} \\ 0 & 0 & V_A k \\ \rho_\tau & \rho_n & 0 \end{vmatrix} = -\bar{\tau}V_A k\rho_n + \bar{n}V_A k\rho_\tau\,. \qquad (11)$$

Considering the fact that the speed of pole A in direction coincides with the orthom $\bar{\tau}$, ie. $\overline{V_A} = V_A\bar{\tau}$, we rewrite expression (8) with regard to (11):

$$\overline{V_B} = V_A\left[\bar{\tau}(1 - k\rho_n) + \bar{n}k\rho_\tau\right]. \qquad (12)$$

The geometric sum of the components (7) and (12) will give the same result:

$$V_B = V_A\sqrt{(1 - k\rho_n)^2 + k^2\rho_\tau^{\,2}}. \qquad (13)$$

Now we consider an alternative with the using of Frenet formulas. The vector equation (3) in the system of the accompanying trihedron will be written as follows:

$$\overline{R_B} = \overline{r_A} + \overline{\tau}\rho_\tau + \overline{n}\rho_n. \tag{14}$$

If we assume that the coordinates ρ_τ and ρ_n do not change along the curve when the trihedron moves, that is, the point *B* is fixed in the trihedron, then its absolute velocity can be found by differentiating expression (14) with respect to time *t*. However, the position of the trihedron on the curve depends on the arc coordinate *s*, so when differentiating (14), it is necessary to go from the independent variable t to the arc s:

$$\frac{d\overline{R_B}}{dt} = \frac{d\overline{R_B}}{ds}\cdot\frac{ds}{dt} = V_A\frac{d\overline{R_B}}{ds} = V_A\left[\frac{d\overline{r_A}}{ds} + \frac{d\overline{\tau}}{ds}\rho_\tau + \frac{d\overline{n}}{ds}\rho_n\right]. \tag{15}$$

In expression (15), the derivative $\frac{d\overline{r_A}}{ds} = \overline{\tau}$, i.e. this is a single unit of the tangent. The remaining derivatives – $\frac{d\overline{\tau}}{ds}$ and $\frac{d\overline{n}}{ds}$ – are known Frenet formulas, which have a kinematic interpretation [6]. They are the basic formulas of differential geometry, in which the independent coordinate is the arc coordinate S (we give a simplified version for a plane curve):

$$\overline{\tau}' = k\overline{n}; \quad \overline{n}' = -k\overline{\tau}, \tag{16}$$

where k – the curvature of the curve, which is given by the natural equation $k = k(s)$.

The Frenet formulas (16) make it possible to quickly and easily obtain derivatives with respect to the arc coordinate s from the unit vectors $\overline{\tau}$ and $\overline{n}$ in the projection on these units. In the kinematic interpretation, the derivatives (16) are projections of the velocities of the ends of unit vectors $\overline{\tau}$ and $\overline{n}$ on these units in the rotational motion of the trihedron [6]. Taking (16) into account, expression (15) assumes exactly the same form as it was obtained in (12). Thus, the application of the Frenet formulas (16) makes it very easy to find the velocity of the point *B* in the rotational motion around the pole *A*, which in the other case must be found as a vector product (11). Even more effectively, they work when finding the absolute speed of point *B* in a complex motion, to the consideration of which we proceed further.

Now we assume that the point *B* moves in the system of the accompanying trihedron, i.e. ector $\overline{\rho}$ is a function of time: $\overline{\rho} = \overline{\rho}(t)$. In this case, the absolute velocity of the point *B* will be determined as the sum of the transport velocity, which can be found from formula (12), and the relative velocity, which we obtain by differentiating the vector $\overline{\rho}$ with respect to time t. However, the portable speed is a function of the arc coordinate S, so the relative speed must also be related to this independent variable:

$$\overline{V_r} = \frac{d\overline{\rho}}{dt} = \frac{d\overline{\rho}}{ds}\frac{ds}{dt} = V_A\frac{d\overline{\rho}}{ds}. \tag{17}$$

Let us write the vector equation (17), which determines the relative velocity in the trihedron system, into two components along the directing vectors $\overline{\tau}$, $\overline{n}$ and we add to the transport velocity (12). After grouping the terms and taking the pole velocity V_A off the brackets, the absolute velocity of the point *B* in the projections onto the vertices of the trihedron can be written as follows:

$$\overline{V}_B = V_A\left[\overline{\tau}\left(1 - k\rho_n + \rho_\tau'\right) + \overline{n}\left(k\rho_\tau + \rho_n'\right)\right]. \tag{18}$$

Now we can show how it is easy to obtain the result (18) with the help of the Frenet formulas. To do this, we differentiate expression (14) under the condition that the coordinates ρ_τ and ρ_n are the functions of the arc coordinate s:

$$\overline{R_B'} = \overline{\tau} + \overline{\tau}'\rho_\tau + \overline{\tau}\rho_\tau' + \overline{n}'\rho_n + \overline{n}\rho_n'. \tag{19}$$

Substituting in (19) the expressions for the orthonormal derivatives of the trihedron from the Frenet formulas (16), we obtain:

$$\begin{aligned}\overline{R_B'} &= \overline{\tau} + nk\rho_\tau + \tau\rho_\tau' - \tau k\rho_n + \overline{n}\rho_n' = \\ &= \overline{\tau}(1 - k\rho_n + \rho_\tau') + \overline{n}(k\rho_\tau + \rho_n').\end{aligned} \tag{20}$$

Comparing equations (18) and (20), we can conclude that the differentiation of equation (14) with the use of Frenet's formulas gives the absolute velocity of the point given in the trihedron system by a removable distance $\overline{\rho} = \overline{\rho}(s)$ in the projections onto the trihedron orbits at the velocity of the trihedron along the curve V_A = 1 m·s^{-1}. In the case where the speed V_A differs from one, each projection must be multiplied by an amount V_A. So, we can formulate the following rule:

If the point in the system of the mobile accompanying trihedron of the curve is given by the radius vector in the form (14), then in order to find its absolute velocity in the projections onto the units of the same trihedron, it is necessary to differentiate equation (14) along the arc coordinate of the curve S using Frenet formulas and to multiply obtained result by the velocity of the vertex of the trihedron along the curve.

The position of the point *B* in the system of the accompanying trihedron of the curve can also be specified in the polar coordinate system – by the distance $\overline{\rho}$ and the angle φ, the count of which is taken from the orbit $\overline{\tau}$ (Fig. 2). In this case, the position of point *B* in the trihedron system in vector form will be written as follows:

$$\overline{R}_B = \overline{r}_A + \overline{\tau}\rho\cos\varphi + \overline{n}p\sin\varphi.$$

To obtain the absolute velocity of the point *B*, we need to differentiate the equation (21) along the arc coordinate s [using the Frenet formula (16) taking into account that $\rho = \rho\ (s)$ and $\varphi = \varphi\ (s)$], and multiply the result by the speed of motion v_A of the top of the trihedron:

$$\begin{aligned}&\overline{V}_B = \overline{V}_A\left\{\overline{\tau}\left[1 + \rho'\cos\varphi - \rho(k + \varphi')\sin\varphi\right] + \right.\\ &\left. + \overline{n}\left[\rho'\sin\varphi + \rho(k + \varphi')\cos\varphi\right]\right\}.\end{aligned} \tag{22}$$

The absolute velocity modulus of the point *B* for formulas (18) and (22) respectively, will be written as follows:

$$V_B = V_A\sqrt{\left(1 - k\rho_n + \rho_\tau'\right)^2 + \left(k\rho_\tau + \rho_n'\right)^2}; \tag{23}$$

$$V_B = V_A\sqrt{\left(\rho' + \cos\varphi\right)^2 + \left[\sin\varphi - \rho\left(k + \varphi'\right)\right]^2}. \tag{24}$$

Now we turn to finding the absolute trajectory of point *B*, i.e. trajectory in the fixed coordinate system *Oxy*. The dependencies $\rho_\tau = \rho_\tau(s)$, $\rho_n = \rho_n(s)$ or $\rho = \rho(s)$, $\varphi = \varphi(s)$ specify the trajectory of motion in the system of the accompanying trihedron, i.e. a trajectory of relative motion. The sum of the relative and movable motions of point *B* will give the trajectory of its absolute motion. After all, we need to go from the vector equations (14) or (21) to their coordinate recording in the projection on the axis of the fixed coordinate system. Due to the motion of the trihedron, the position of its vertex $A(x_A, y_A)$ in the *Oxy* system will vary depending on the arc coordinate s. The coordinates of vertex *A* in the projections on the axis of the fixed system *Oxy* can be found if there is known the dependence $k = k(s)$ – the so-called natural equation of the curve. The transition formulas have the form [7]:

$$x_A = \int\cos\ \alpha\ ds; \qquad y_A = \int\sin\alpha\ ds, \tag{25}$$

where $\alpha = \alpha(s)$ – the regularity of the angle α change (Fig. 2) when the vertex *A* moves along the curve and which is also determined from the dependence $k = k(s)$ [7]:

$$\alpha = \int k ds. \tag{26}$$

The absolute trajectory of the point *B* in the coordinate system *Oxy* is obtained by parallel transfer of the vertex *A* along the axes on the parameters (25) and by the simultaneous transition from the coordinates of the point *B* (ρ_τ, ρ_n or $\rho\cos\varphi$, $\rho\sin\varphi$) in the trihedron system to the coordinates of the point $B(x_B, y_B)$ in the fixed coordinate system. To do this, we combine their axes by turning the trihedron around the binormal by an angle $\alpha = \alpha(s)$.

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After the rotation, summation and substitution (26) in (25), the vector equation (14) is written in the projections on the axis of the fixed coordinate system:

$$\begin{aligned} x_B &= \rho_\tau \cos\left(\int kds\right) - \rho_n \sin\left(\int kds\right) + \int \cos\left(\int kds\right) ds; \\ y_B &= \rho_\tau \sin\left(\int kds\right) + \rho_n \cos\left(\int kds\right) + \int \sin\left(\int kds\right) ds. \end{aligned} \tag{27}$$

Since the parametric equations (27) describe the absolute trajectory, their differentiation with respect to a parameter s can be used to find the components and the modulus of the absolute velocity of point B as a function of s :

$$\begin{aligned} x'_B &= (1 - k\rho_n + \rho'_\tau)\cos\left(\int kds\right) - (k\rho_\tau + \rho'_n)\sin\left(\int kds\right); \\ y'_B &= (1 - k\rho_n + \rho'_\tau)\sin\left(\int kds\right) + (k\rho_\tau + \rho'_n)\cos\left(\int kds\right); \\ V_B(s) &= \sqrt{x'^2_B + y'^2_B} = \sqrt{(1 - k\rho_n + \rho'_\tau)^2 + (k\rho_\tau + \rho'_n)^2}\,. \end{aligned} \tag{28}$$

Comparing the last expression (28) and expression (23), we conclude that they are similar. For a complete analogy, the last expression (28) needs to be multiplied by the speed V_A, since in this case we will move from the equation $V_B = V_B(s)$ to the equation $V_B = V_B(t)$.

Applying a similar coordinate transformation with respect to the vector equation (21), we obtain the parametric equations of the absolute trajectory of the point *B*:

$$\begin{aligned} x_B &= \rho\cos\varphi\cos\left(\int kds\right) - \rho\sin\varphi\sin\left(\int kds\right) + \int \cos\left(\int kds\right) ds; \\ y_B &= \rho\cos\varphi\sin\left(\int kds\right) + \rho\sin\varphi\cos\left(\int kds\right) + \int \sin\left(\int kds\right) ds. \end{aligned} \tag{29}$$

Equation (29) can be written in a more compact form by applying trigonometric formulas for the sum and difference of angles:

$$\begin{aligned} x_B &= \rho\cos\left(\varphi + \int kds\right) + \int \cos\left(\int kds\right) ds; \\ y_B &= \rho\sin\left(\varphi + \int kds\right) + \int \sin\left(\int kds\right) ds. \end{aligned} \tag{30}$$

Analogously to the previous case, by differentiating (30) with respect to the parameter s, we can find the absolute velocity components of the point B $V_B = V_B(s)$, multiplying which by V_A we obtain the result which was obtained earlier in (24).

We consider the following examples. In view of the directing (initial) curve along which the accompanying trihedron moves with speed V_A , we take a chain line whose natural equation has the form:

$$k = \frac{a}{a^2 + s^2}, \tag{31}$$

where a – constant parameter.

The absolute velocity can be found from formulas (18) and (23) and an absolute trajectory according to equations (27) in the case when the relative motion in the trihedron is given by the dependences of $\rho_\tau = \rho_\tau(s)$ and $\rho_n = \rho_n(s)$. If these dependences have the form $\rho = \rho(s)$ and $\varphi = \varphi(s)$, then we must use formulas (22), (24) and (30). We will use the second variant and consider the kinematics of the point B for some dependences $\rho = \rho(s)$ and $\varphi = \varphi(s)$. After substituting (21) into (30) and by integrating, we obtain:

$$\begin{aligned} x_B &= \frac{\rho}{\sqrt{a^2 + s^2}}\left(a\cos\varphi - s\sin\varphi\right) + a\mathrm{Arsh}\frac{s}{a}; \\ y_B &= \frac{\rho}{\sqrt{a^2 + s^2}}\left(a\sin\varphi + s\cos\varphi\right) + \sqrt{a^2 + s^2}\,. \end{aligned} \tag{32}$$

By substituting the given dependences $\rho = \rho(s)$ and $\varphi = \varphi(s)$ in (32), we obtain the parametric equations of the absolute trajectory of the point B. We find the absolute velocity in the projections onto the unit vectors of the accompanying trihedron from expression (22), and its modulus from expression (24). On Fig. 3 from the equations (32) there are constructed the absolute trajectories of the point B for different dependences of $\rho = \rho(s)$ and $\varphi = \varphi(s)$.

The value of the constant a is assumed to be a = 25, the change in the arc coordinate s occurred within the range s = 0 – 100. For ρ = 0, from equations (32) we obtain the initial curve – the chain line, which is on Fig. 3 depicted by thickened line. On Fig. 4 there are presented the graphs of the absolute velocity modulation of the point B, which are plotted as a function of the arc coordinate s, using formula (24) for V = 1 m·s^{-1} for the trajectories shown in Fig. 3c and in Fig. 3d.

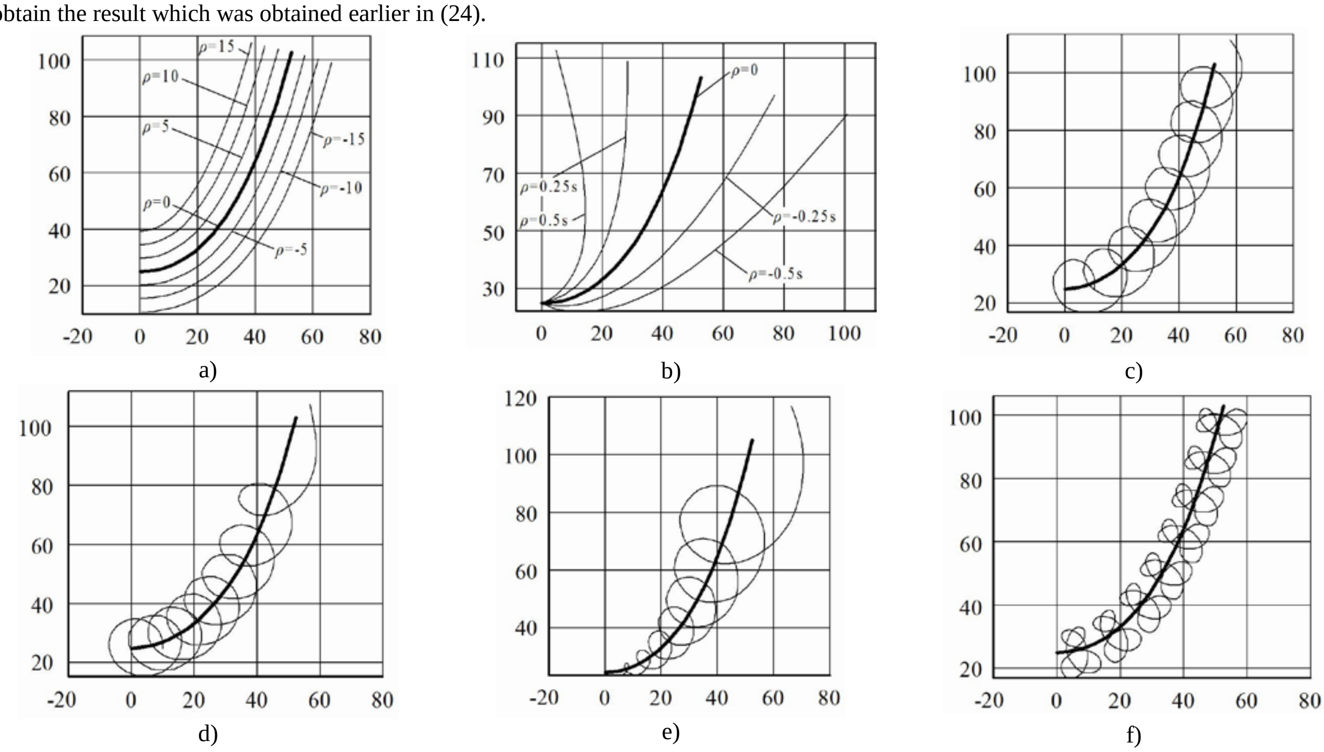
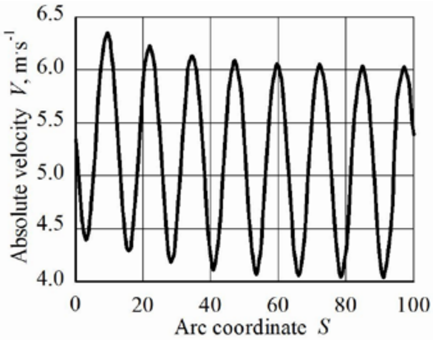
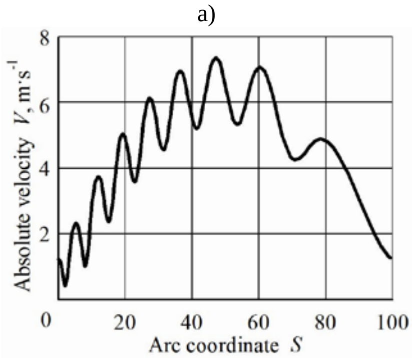


***Fig. 3**. – Absolute trajectories of the point B for different dependences $\rho = \rho(s)$ and $\varphi = \varphi(s)$, constructed from equations (32):*
a) φ = 90° – const ; ρ = const (trajectories – equidistant curves); b) φ = 90° – const ; ρ = cs – it changes linearly;
c) φ = 0,5 s ; ρ = 10 – const; d) φ = s – 0,005 s^2; ρ = 10 – const; e) φ = s – 0,005 s^2; ρ = 0,25 s; f) φ = 0,5 s; ρ = 10 sin s



a)



b)

***Fig. 4.** – The graphs of the changes of modulus of the absolute velocity of the point B in relation to the arc coordinate s for V = 1 m·s-1: a) the graph of the changes of the velocity of the point for its absolute trajectory shown on Fig.3d; b) the graph of the change in the velocity of a point for its absolute trajectory, shown in Fig. 3e*

Let us proceed to the next stage – finding the absolute acceleration of point *B*. According to the classical theory, it is defined as the geometric sum of the three vectors:

$$\overline{w_B} = \overline{w_e} + \overline{w_r} + 2\overline{\omega} \times \overline{V_r} \ . \tag{33}$$

The first vector from (33) is called the transport acceleration and is determined by formula [4, 7]:

$$\overline{w_e} = \overline{w_A} + \overline{\varepsilon} \times \overline{\rho} + \overline{\omega} \times (\overline{\omega} \times \overline{\rho}), \tag{34}$$

where $\overline{\varepsilon}$ – vector of the angular acceleration.

We find the expressions for all the components of expression (34) and their sum. The first vector $\overline{w_A}$ of acceleration of the origin of the trihedron it is found by differentiating the corresponding velocity, while moving from the time parameter t to the arc coordinate s:

$$\begin{aligned} \overline{w_A} &= \frac{d\overline{V_A}}{dt} = \frac{d\overline{V_A}}{ds} \cdot \frac{ds}{dt} = V_A \frac{d\overline{V_A}}{ds} = \\ &= V_A \frac{d}{ds}\left(V_A \overline{\tau}\right) = v_A \left(\frac{dV_A}{ds}\overline{\tau} + V_A \frac{d\overline{\tau}}{ds} \right). \end{aligned} \tag{35}$$

The equation (35) $\dfrac{d\overline{\tau}}{ds}$ can be considered as first from the frenet formulas (16). Taking into account (16) the equation (35) can be written as follows:

$$\overline{w_A} = V_A V_A' \overline{\tau} + V_A^2 k \overline{n} \ . \tag{36}$$

In the case when $v_A = const$ the acceleration of the vertex of the trihedron will have one component directed along the principal normal $\overline{n}$, and its modulus will have a value $V_A^2 \cdot k$ or V_A^2 / r, since $k = 1/r$, where r – is the radius of curvature of the curve. This is known as the so-called normal acceleration. If the speed of motion of the trihedron is variable, then another component appears, directed along the tangent – tangential acceleration.

Thus, expression (36) is a well-known formula for determining the acceleration of a point's motion along a curve in which an arc coordinate s serves instead of a time variable. The second component of the (34) includes the angular acceleration vector $\overline{\varepsilon}$. To determine it, we differentiate the angular velocity vector $\overline{\omega}$. According to (2) we know the parameter ω: $\omega = V_A k$. Since the vector $\overline{\varepsilon}$ is directed along the binormal $\overline{b}$, the differentiation gives:

$$\begin{aligned} \overline{\varepsilon} &= \frac{d\overline{\omega}}{dt} = \frac{d\overline{\omega}}{ds}\frac{ds}{dt} = V_A \frac{d\overline{\omega}}{ds} = \\ &= V_A \frac{d}{ds}\left(\overline{b} V_A k\right) = v_A \left[\overline{b}' V_A k + \overline{b}(V_A k)'\right]. \end{aligned} \tag{37}$$

In view of the Frenet formulas (16) and after further differentiation, we obtain:

$$\overline{\varepsilon} = \overline{b} \cdot V_A \left(V_A' \cdot k + V_A \cdot k'\right). \tag{38}$$

Now we find a vector composition $\overline{\varepsilon} \times \overline{\rho}$:

$$\begin{aligned} \overline{\varepsilon} \times \overline{\rho} &= \begin{vmatrix} \overline{\tau} & \overline{n} & \overline{b} \\ 0 & 0 & V_A(V_A' \cdot k + V_A' \cdot k') \\ \rho_\tau & \rho_n & \rho_b \end{vmatrix} = \\ &= -\overline{\tau} \cdot V_A \cdot \rho_n (V_A' \cdot k + V_A \cdot k') + \overline{n} \cdot V_A \cdot \rho_\tau (V_A' \cdot k + V_A \cdot k'). \end{aligned} \tag{39}$$

The last component in expression (34) - vector composition $\overline{\omega} \times (\overline{\omega} \times \overline{\rho})$ we can find by analogous way. Below is the final result:

$$\overline{\omega} \times \overline{\rho} = -\overline{\tau} \cdot \rho_n \cdot v_A \cdot k + \overline{n} \cdot \rho_\tau \cdot v_A \cdot k. \tag{40}$$

$$\overline{\omega} \times (\overline{\omega} \times \overline{\rho}) = -\overline{\tau} \cdot V_A^2 \cdot k^2 \cdot \rho_\tau - \overline{n} \cdot V_A^2 \cdot k^2 \cdot \rho_n. \tag{41}$$

Substituting the vectors (36), (39) and (41) into (34), after grouping of the components according to corresponding directions of the unit vectors, we obtain the vector of the transport acceleration:

$$\begin{aligned} \overline{w_e} = &\overline{\tau} V_A \left[V_A' - \rho_n (V_A' k + V_A k') - V_A k^2 \rho_\tau\right] + \\ &+ \overline{n} V_A \left[V_A k + \rho_\tau (V_A' k + V_A k') - V_A k^2 \rho_n\right]. \end{aligned} \tag{42}$$

The next component in the formula (33) is called the relative acceleration, i.e. this is the acceleration of point *B* with respect to the system of the Frenet trihedron. It can be obtained by differentiating the expression for the relative velocity. The relative velocity V_r is obtained as the derivative of the radius vector $\overline{\rho}$ in the system of the accompanying trihedron:

$$\overline{V_r} = \frac{d\overline{\rho}}{dt} = \frac{d\overline{\rho}}{ds} \cdot \frac{ds}{dt} = V_A \frac{d\overline{\rho}}{ds}. \tag{43}$$

After differentiating of the expression (43), we obtain:

$$\overline{w_r} = \frac{d}{dt}\left(V_A \frac{d\overline{\rho}}{ds}\right) = \frac{ds}{dt} \cdot \frac{d}{ds}\left(V_A \frac{d\overline{\rho}}{ds}\right) = V_A (V_A' \overline{\rho}' + V_A \overline{\rho}''). \tag{44}$$

Placing the vector (44) along the directions of the unit vectors of the trihedron, we obtain:

$$\overline{w_r} = V_A \left[\overline{\tau}(V_A' \rho_\tau' + V_A \rho_\tau'') + \overline{n}(V_A' \rho_n' + V_A \rho_n'')\right].$$

Finally, the third, the last vector in the expression (33) is called the Coriolis acceleration. We find it as a doubled vector conjuction of the angular velocity vector $\overline{\omega} = \overline{b} \cdot V_A \cdot k$ and relative velocity vector V_r (43). We will have:

$$2\overline{\omega} \times \overline{V_r} = \begin{vmatrix} \overline{\tau} & \overline{n} & \overline{b} \\ 0 & 0 & V_A k \\ V_A \rho_\tau' & V_A \rho_n' & 0 \end{vmatrix} = 2V_A^2 k(-\overline{\tau}\rho_n' + \overline{n}\rho_\tau'). \tag{46}$$

Substituting (42), (45) and (46) into (33) and grouping the components of the vectors along the directions of the unit vectors of the trihedron, we finally obtain the expression for absolute acceleration of point *B*:

$$\begin{aligned} \overline{w_B} = &\overline{\tau} V_A \left[V_A'(1 - k\rho_n + \rho_\tau') + V_A(\rho_\tau'' - k'\rho_n - k^2 \rho_\tau - 2k\rho_n') \right] + \\ &+ \overline{n} V_A \left[V_A'(k\rho_\tau + \rho_n') + V_A(\rho_n'' - k'\rho_\tau - k^2 \rho_n + k + 2k\rho_\tau') \right]. \end{aligned} \tag{47}$$

By formula (47), we can find the absolute acceleration of point *B* in complex motion if there known the law of its motion in the contiguous plane of the trihedron $\rho_\tau = \rho_\tau(s)$, $\rho_n = \rho_n(s)$, and the trihedron itself moves with a given velocity $v = v(s)$ along a plane curve with the known natural equation $k = k(s)$. It should be emphasized that the absolute acceleration is obtained in the projections on the axis of the mobile accompanying trihedron of the curve.

Now we find the formula for the absolute acceleration of point *B* when its motion is given by equation (21), that is:

$$\rho_\tau = \rho\cos\varphi; \qquad \rho_n = \rho\sin\varphi. \tag{48}$$

Differentiating twice the equations (48), we obtain:

$$\begin{aligned}
&\rho'_\tau = \rho'\cos\varphi - \rho\varphi'\sin\varphi;\\
&\rho'_n = \rho'\sin\varphi + \rho\varphi'\cos\varphi;\\
&\rho''_\tau = \left(\rho'' - \rho\varphi'^2\right)\cos\varphi - \left(2\rho'\varphi' + \rho\varphi''\right)\sin\varphi;\\
&\rho''_n = \left(\rho'' - \rho\varphi'^2\right)\sin\varphi + \left(2\rho'\varphi' + \rho\varphi''\right)\cos\varphi.
\end{aligned} \tag{49}$$

Substituting (49) in (47) we obtain an expression for finding the absolute acceleration of point *B* in the case when its relative motion is specified by the distance $\rho = \rho(s)$ and angle $\varphi = \varphi(s)$:

$$\begin{aligned}
\overline{w}_B = {} & \overline{\tau}V_A\left\{V'_A\left[1 + \rho'\cos\varphi - \rho(k+\varphi')\sin\varphi\right] + \right.\\
& + V_A\left[\rho'' - \rho(k+\varphi')^2\right]\cos\varphi -\\
& \left. -\left[2\rho'(k+\varphi') + k'\rho + \rho\varphi''\right]\sin\varphi\right\} +\\
& + \overline{n}V_A\left\{V'_A\left[\rho'\sin\varphi + \rho(k+\varphi')\cos\varphi\right] + \right.\\
& + V_A\left[\rho'' - \rho(k+\varphi')^2\right]\sin\varphi +\\
& \left. + \left[2\rho'(k+\varphi') + k'\rho + \rho\varphi''\right]\cos\varphi + k\right\}.
\end{aligned} \tag{50}$$

The modulus of the absolute acceleration vector of the point *B* (47), which is given by the projections onto the orthograms of the trihedron, or (50), where the point *B* is given by the distance *ρ* and the angle *φ*, is defined as the geometric sum of its projections on the unit vectors $\overline{\tau}$ и $\overline{n}$:

$$w_B = \sqrt{\overline{w}_{B\tau}^{\,2} + \overline{w}_{Bn}^{\,2}}. \tag{51}$$

Formulas (47), (50) for finding of the absolute acceleration are obtained by methods of the classical theory with finding each component: transport acceleration, relative acceleration and acceleration of Coriolis.

And now we show how simply can be obtained theses formulas with the help of Frenet formulas, without dwelling on finding each individual component of absolute acceleration. This is the purpose of this study.

The determination of the vecor of an absolutte acceleration of the point *B* is carried out by differentiating the expressions (18) or (22) of the absolute velocity, since this is done in the study of ordinary motion. Anyway, it must be differentiated along the arc coordinate s, since the expressions (18), (22) are its functions. We differentiate, for example, expressions (18) in detail using the Frenet formulas (16):

$$\begin{aligned}
\overline{V'_B} = {} & V'_A\left[\overline{\tau}(1 - k\rho_n + \rho'_\tau) + \overline{n}(k\rho_\tau + \rho'_n)\right] +\\
& + V_A\left[\overline{\tau}(1 - k\rho_n + \rho'_\tau) + \overline{n}(k\rho_\tau + \rho'_n)\right]' =\\
= {} & V'_A\left[\overline{\tau}(1 - k\rho_n + \rho'_\tau) + \overline{n}(k\rho_\tau + \rho'_n)\right] +\\
& + V_A\left[\overline{\tau}'(1 - k\rho_n + \rho'_\tau) + \overline{\tau}(1 - k\rho_n + \rho'_\tau)' + \right.\\
& \left. + \overline{n}'(k\rho_\tau + \rho'_n) + \overline{n}(k\rho_\tau + \rho'_n)'\right] =\\
= {} & V'_A\left[\overline{\tau}(1 - k\rho_n + \rho'_\tau) + \overline{n}(k\rho_\tau + \rho'_n)\right]\\
& + V_A\left[\overline{n}k(1 - k\rho_n + \rho'_\tau) + \overline{\tau}(-k'\rho_n - k\rho'_n + \rho''_\tau) - \right.\\
& \left. - \overline{\tau}k(k\rho_\tau + \rho'_n) + \overline{n}(k'\rho_\tau + k\rho'_\tau + \rho''_n)\right].
\end{aligned} \tag{52}$$

Grouping in the expression (52) the components along the directions of the unit vectors $\overline{\tau}$ and $\overline{n}$, we obtain:

$$\begin{aligned}
\overline{V'_B} = {} & \overline{\tau}\left[V'_A(1 - k\rho_n + \rho'_\tau) + V_A(\rho''_\tau - k'\rho_n - k^2\rho_\tau - 2k\rho'_n)\right] +\\
& + \overline{n}\left[V'_A(k\rho_\tau + \rho'_n) + V_A(\rho''_n + k'\rho_\tau - k^2\rho_n + k + 2k\rho'_\tau)\right].
\end{aligned}$$

Comparing the expressions (47) and (53), we see that they differ only by multiplier V_A. This is understandable, since we have differentiated the expression (18) along the arc coordinate s. When differentiating with respect to time t, as it is necessary to do in order to find the acceleration, we obtain

$$\overline{w}_B = \frac{d\overline{V}_B}{dt} = \frac{d\overline{V}_B}{ds}\frac{ds}{dt} = V_A\frac{d\overline{V}_B}{ds}, \tag{54}$$

i.e. from (54) it is clear that the result (53) obtained must be multiplied by the velocity V_A. After this, expression (53) will be analogous to expression (47). In the same way, by differentiating expressions (22) one can obtain an expression (50).

So, it was shown how simply is to find the absolute acceleration vector of point *B* in complex motion with the application of the accompanying trihedron of the portable trajectory and the Frenet formulas. The obtained result can be formulated in the form of the following rule:

If the material point in the system of the mobile accompanying trihedron of the curve is given by the radius vector in the form (14), in order to find its absolute acceleration in the projections onto the units of the same trihedron, it is necessary to differentiate the expression of the absolute velocity (18) along the arc coordinate s using the Frenet formulas and the obtained result to multiply by the velocity of the vertex of the trihedron along the curve.

The formulated rule also applies to formula (21), when the material point in the contiguous plane of the trihedron is described as a polar coordinate system, i.e. it is necessary to differentiate expression (22) and obtained result to multiply by the velocity of the vertex of the trihedron along the curve.

Let us consider an example that explains the dynamics of the motion of a material point in a complex motion.

A tractor trailer that contains a flat cargo moves at a constant speed V_A along a curve, which is a chain line given by the natural equation (31). At a certain point of time, as the curvature of the curve increases, it comes into motion relative to the trailer. To find the relative and absolute trajectories of cargo movement, as well as its speed, if the location of the cargo in the trailer at the beginning of the slip and the coefficient of friction *f* are known.

Neglecting the size of the cargo, we take it for the material point, which is in the front left corner of the trailer along the tractor movement. This angle is taken as the vertex of the trihedron, which is rigidly tied to the trailer, and the ort $\overline{\tau}$ is directed along the tangent to the chain line along which the indicated point of the trailer moves, and the unit vector $\overline{n}$ – to the center of curvature of the curve. The parametric equations of the chain line after the transition from the natural equation to the parametric equations according to (25), (26) take the following form:

$$x = a\,\mathrm{Arsh}\frac{s}{a}; \qquad y = \sqrt{a^2 + s^2}. \tag{55}$$

This curve has an axis of symmetry that passes through the vertex (at $s = 0$), in which the curvature is the largest and takes the value $k = 1 / a$. When the trihedron moves along a curve with a constant velocity in the direction of the vertex, the curvature of the chain line will increase, like the centrifugal force. In this case, the moment may come when the frictional force will be overcome and the relative movement of the load in the trihedron (or trailer) system will begin.

To compose the equation of motion in the form $m\overline{w_B} = \overline{F}$, we must find the expression for the absolute acceleration of the particle *B*. We obtain it from (47) for $V'_A = 0$. Since the applied friction force $F = f\,mg$ acts in the direction opposite to the relative velocity, it is necessary to find the projections of the unit tangent vector to the relative trajectory. Its projection on the unit vectors $\overline{\tau}$ and will $\overline{n}$ have the same ratio, which is the components of the relative

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velocity ρ'_τ и ρ'_n т.е.:

$$\frac{\rho'_\tau}{\sqrt{\rho'^2_\tau + \rho'^2_n}} \quad \text{and.} \quad \frac{\rho'_n}{\sqrt{\rho'^2_\tau + \rho'^2_n}} \tag{56}$$

Let us write down the vector equation $m\overline{w_B} = \overline{F}$ in the projections onto the unit vectors of the trihedron, taking (50) and (56) into account, and also that $V = const$. After reducing to the mass m particles, we obtain a system of two differential equations in the form:

$$\begin{aligned} &V_A^2(\rho''_\tau - k'\rho_n - k^2\rho_\tau - 2k\rho'_n) = -fg\cdot\frac{\rho'_\tau}{\sqrt{\rho'^2_\tau + \rho'^2_n}}; \\ &V_A^2(\rho''_n + k + k'\rho_\tau - k^2\rho_n + 2k\rho'_\tau) = -fg\cdot\frac{\rho'_n}{\sqrt{\rho'^2_\tau + \rho'^2_n}}, \end{aligned} \tag{57}$$

where $k = \dfrac{a}{a^2+s^2}$; $k' = -\dfrac{2as}{\left(a^2+s^2\right)^2}$.

A graphical presentation of the results of numerical integration of the system (57) is shown in Fig. 5. Integration was carried out by changing the arc coordinate s from –80º to +80º. The value of the constants is: a = 25; f = 0.35; V = 10 m·s^{-1}.

From Fig. 5a, it is seen that the relative movement of the load began at approximately $s \approx -10$º and it is ended at $s \approx 25$º, with the maximum relative velocity reaching $V_r \approx 0.8$ m·s^{-1}.

The value of the arc coordinate s, at which the relative motion began, can also be determined analytically. Relative motion will begin when the centrifugal force at the vertex of the trihedron (i.e., with $\rho_\tau = \rho_n = 0$) exceeds the frictional force $F_m = f\,mg$. Equating these forces and substituting the expression $k = k(s)$, we obtain an equation with an unknown value of the arc coordinate s of the form:

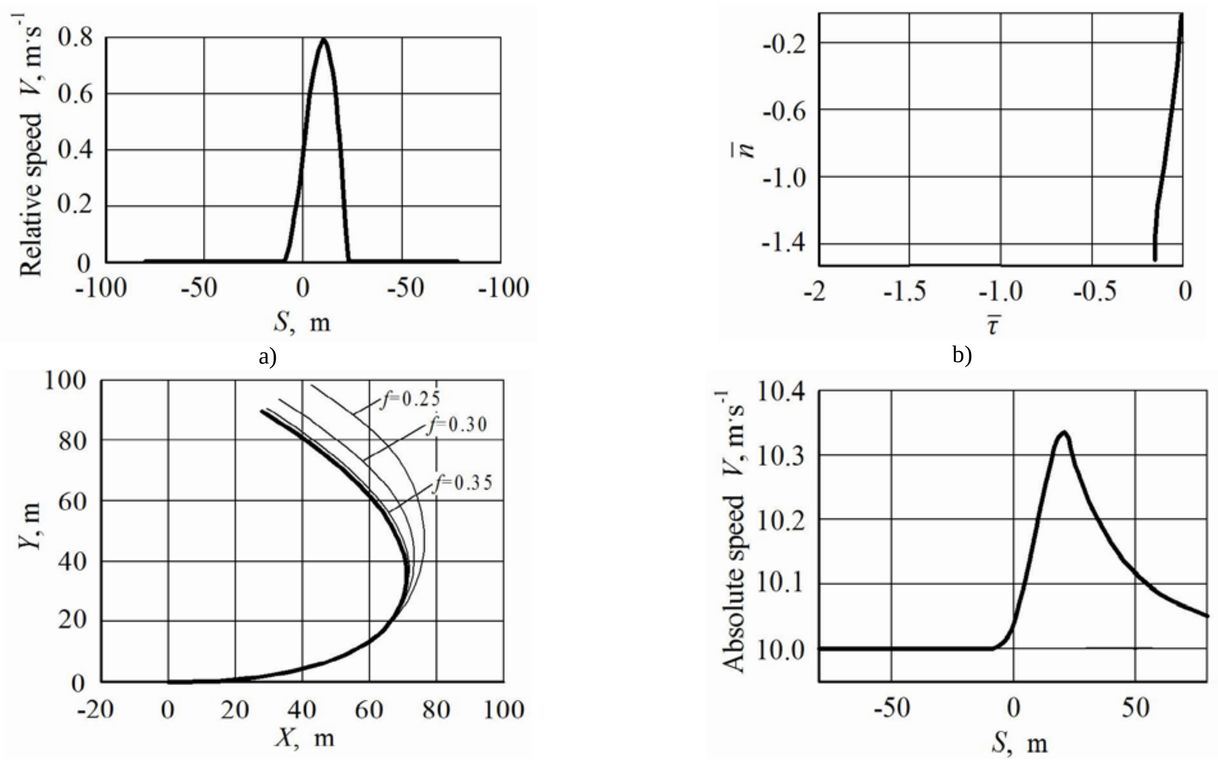
$$\frac{m\,V_A^2\,a}{a^2+s^2} = f\,mg\ ,$$

from where

$$s = \sqrt{\frac{a}{fg}\left(V_A^2 - afg\right)}\,. \tag{58}$$

The solution of equation (58) for conditions of the indicated constants shows that the relative movement of the load begins at s > –10.15°. Having passed the way to the symmetrical point at s = 10.15º, the cargo continues to move along the body, but with a slowdown, since the values of the centrifugal force are not sufficient to continue this movement.

The graph of the relative motion trajectory (Figure 5, b) shows that the load in the trailer will move approximately 1.5 ° towards the opposite side and approximately 0.2 ° in the direction opposite to the tractor's direction of travel.



c)
d)

***Fig. 5**. Graphs of dependencies obtained as a result of integration of the system (57):*
a) graph of the change of relative speed; b) the trajectory of relative motion in the trihedron system;
c) a chain line and absolute trajectories of motion (additionally shown for f = 0.3 and f = 0.25); d) graph of absolute speed changes

The graphs of the absolute trajectory (Figure 5, c) show that for different coefficients of friction the relative movement of cargo in the trailer starts from different points of the chain line. After the discontinuance of the relative movement, the absolute path of the load obtains the form of a curve parallel to the chain line.

From the graph of the change in absolute velocity (Figure 5, d) it can be seen that after the discontinuance of a relative movement, the absolute speed of the load will be greater than it was before it, since it occupies another position in the trihedron system.

There are possible also other examples of similar processes.

4. Conclusions

The application of the accompanying trihedron of a plane curve as a moving coordinate system relative to which a relative motion of the point is carried out is quite possible when investigating the complex motion of a material point along the plane. The Frenet formulas make it possible to quickly and easily find the absolute velocity of a material point in its complex motion in the projections onto the orthograms of the trihedron and to find the absolute trajectory in a fixed coordinate system.

In this case, it is much easier to find the absolute acceleration

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of a point in a complex motion in the projections onto the orthogonal faces of the trihedron, which automatically includes all three of its components. This allows us to solve, on a new scale, the problems of the dynamics of a material point in the moving system of the Frenet trihedron. A method which is developed considerably simplifies the solution of problems of complex motion of a material point, which determines its further development and effective application.

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