International Workshop on the Qualitative Theory of Differential Equations

QUALITDE – 2023

Dedicated to the 120th birthday anniversary of Professor V. Kupradze

December 9 – 11, 2023, Tbilisi, Georgia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University
2, Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Program

December 9, 2023

13:45 – 14:00 Opening of the Workshop

14:00 – 14:30 **I. Kiguradze** – Blow-Up Solutions of the Cauchy Problem for Nonlinear Delay Ordinary Differential Equations

14:30 – 15:00 **J. Šremr** – Parameter-Dependent Periodic Problems for Non-Autonomous Duffing Equations with a Sign-Changing Forcing Term

15:00 – 15:30 **S. Kharibegashvili** – Antiperiodic Problem for One Class of Nonlinear Partial Differential Equations

15:30 – 16:00 **N. Partsvania** – Rapidly Growing Solutions to Two-Dimensional Nonlinear Differential Systems

16:00 – 16:30 C o f f e e B r e a k

16:30 – 19:00 Overview of the talks of participants in absentia

J. López-Gómez, P. Omari – Branches of Positive Solutions of a Superlinear Indefinite Prescribed Curvature Problem

C. Wang, R. P. Agarwal – General Theory of the Higher-Order Linear Quaternion q-Difference Equations

C. Mesquita, M. Tvrdý – On the Existence of Bifurcation Points for Periodic Problems to Distributional Differential Equations

V. Mikhailets, A. Goriunov, V. Molyboga – Sturm-Liouville Operators with Strongly Singular Coefficients: Semi-Boundedness and Self-Adjointness

E. Bravyi – On Solvability Conditions for the Cauchy Problem for Second Order Linear Non-Volterra Functional Differential Equations

A. Lomtatidze – Periodic Solutions of a Pendulum Like Planar System

Z. Došlá, M. Marini, S. Matucci – Asymptotic Proximity Between Equations with Mean Curvature Operator and Linear Equation

S. Staněk – Solvability of BVPs for Sequential Fractional Differential Equations at Resonance

T. Tanigawa – Existence and Asymptotic Behavior of Nonoscillatory Solutions of Quasilinear Differential Equations with Variable Exponents

J. Jaroš, T. Kusano – Nonoscillation Theory of Nonlinear Differential Equations of Emden-Fowler Type with Variable Exponents

E. A. Barabanov, V. V. Bykov – Dimensions of Subspaces Defined by the Lyapunov Exponents of Regular Linear Differential Systems with Parametric Perturbations Vanishing at Infinity as Functions of the Parameter

A. V. Lipnitskii – On the Instability of Millionshchikov Linear Systems with Smooth Dependence on a Parameter

A. N. Vetokhin – Some Properties of Topological Entropy of Families of Dynamical Systems on the Cantor Set

E. Musafirov, A. Grin, A. Pranevich – On Admissible Perturbations of 3D Autonomous Polynomial ODE Systems

December 11, 2023

10:00 – 12:30 Overview of the talks of participants in absentia

M. Perestyuk, V. Slyusarchuk – The Method of Local Linear Approximation in the Theory of Nonlinear Impulse Systems

K. Mamsa, Yu. Perestyuk, M. Perestyuk – On the Stability of Toroidal Manifold for One Class of Dynamical System

O. Kapustyan, A. Krasnieieva – Stability of Global Attractors for the Chafee-Infante Equation w.r.t. Boundary Disturbances

O. Stanzhytskyi, V. Tsan, O. Martynyuk – Periodic Solutions in Dynamic Equations on Time Scales and Their Relationship with Differential Equations

L. Roksolana, V. Mogylova, T. Koval'chuk – The Optimal Control Problem of a System of Integro-Differential Equations on Infinite Horizon

V. Sobchuk, I. Zelenska – A System of Singularly Perturbed Differential Equations with an Unstable Turning Point of the First Kind

A. K. Demenchuk, A. V. Konyukh – On the Massera's Theorem of Existence of Periodic Solutions of Linear Differential Systems

O. Perehuda, F. Asrorov – Investigation of the Behavior of Solutions of Stochastic Ito Differential Equations

V. V. Karapetrov – Solutions of Some Type n-th Order Differential Equations

P. Feketa, J. Fedorenko, D. Bezushchak, A. Sukretna – Qualitative Behavior of the Trajectories Impulsive Semigroup for the Hyperbolic Equation

I. V. Astashova, G. A. Chechkin, A. V. Filinovskiy, T. A. Shabatina, Yu. Morozov – General Decreasing Solutions to the Equation Arises in Cryochemistry

V. M. Evtukhov, S. V. Golubev – Asymptotic Behaviour of Solutions of One Class of Nonlinear Differential Equations of Fourth Order

O. Pravdyvyi, V. Kravets – Existence an Uniqueness of Weak Solutions of Stochastic Functional-Differential Neutral Equations in Hilbert Space

M. I. Zaidel – Stability and Exponential Stability Indices of a Linear System Depending on a Parameter

Existence an Uniqueness of Weak Solutions of Stochastic Functional-Differential Neutral Equations in Hilbert Space

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We consider the following stochastic functional-differential neutral equation on Hilbert space:

$$d(u(t) - g(u_t)) = (f(u_t) + Au)\,dt + \sigma(u_t)\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t) = \phi(t), \quad t \in [-h, 0], \tag{0.2}$$

where

- $u_t = u(t + \theta)$, $\theta \in [-h, 0]$;
- A – linear operator on separable Hilbert space H;
- $W(t)$ – Q-Wiener process on separable Hilbert space K;
- $u(t)$ – state process;
- f – functional from $C([-h, 0], H)$ into H;
- σ – mapping from same space to special space of Hilbert–Smidt operators;
- $\phi : [-h, 0] \to H$ – initial condition,

while existence and uniqueness of a mild solution of the given equation (0.1), (0.2) is known, weak solutions is relatively undiscovered field.

Thus, we consider existence of weak solutions of equation (0.1), (0.2).

1 Preliminaries

Let's assume that K and H are Hilbert spaces, and V, V' is such Banach spaces that

$$V \subset H = H' \subset V'$$

is a Gelfand triple.

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $\{F_t; t \geq 0\}$ generated by the Q-Wiener process W on (Ω, F, P) with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete ortonormal system e_k in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$, and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion $W(t)=\sum\limits_{k=1}^{\infty}\lambda_k\beta_k(t)e_k$, where $\beta_k(t)$ are real valued Brownian motions mutually independent on (Ω,F,P).

Let $U_0=Q^{\frac{1}{2}}(U)$ and $L_0^2=L_2(U_0,H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi,\Psi)L_0^2=\operatorname{tr}[\Phi Q\Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

$C:=C([-h,0];H)$ is the space of continuous mappings from $[-h,0]$ to H equipped with the norm $\|u\|_C=\sup\limits_{\theta\in[-h,0]}\|u(\theta)\|$, and $L_2^V:=L_2((-h,0);V)$ is the space of V-valued mappings with the norm

$$\|u\|_V^2:=\int\limits_{-h}^{0}\|u(t)\|_V^2\,dt.$$

2 Conditions on functions

To ensure existence and uniqueness of a solution, we have to impose additional conditions on functions A, f, σ, g.

Conditions on A:

(A1) Domain of A - $D(A)$ is dense in H such that $A:V\to V'$;

(A2) For any $u,v\in V$ there exist $\alpha>0$:

$$|\langle Au,v\rangle|\le\alpha\|u\|_V\|v\|_V;$$

(A3) A satisfies the coercitivity condition: $\exists\,\beta>0$, γ:

$$\langle Av,v\rangle\le-\beta\|v\|_V^2+\gamma\|v\|_V^2,\ \ \forall\,v\in V.$$

Conditions on g:

(G1) g are mapping from $C\cap L_V^2$ to H;

(G2) (Growth condition) $\exists\,K>0$:

$$\|g(\phi)\|_V^2\le K(1+\|\phi\|_{L_2^V}^2),\ \ \forall\,\phi\in L_2^V;$$

(G3) (Lipshitz condition) $\exists 1/2>L>0$:

$$\|g(\phi)-g(\psi)\|_V\le L\|\phi(t)-\psi(t)\|_V,\ \ \forall\,t\in V.$$

Composite conditions:

(C1) f is a mapping from $C\cap L_V^2$ to H, σ is a mapping from $C\cap L_V^2$ to L_2^0;

(C2) (Growth condition) There $\exists\,K>0$, $\theta\ge1$:

$$\|f(\phi)\|_V\le K\bigg(1+\Big(\int\limits_{-h}^{0}\|\phi(t)\|_V\,dt\Big)^{\theta}+\|\phi\|_V^{\theta}\bigg)$$

and

$$\|\sigma(\phi)\|_{L_2^0}^2\le K\big(1+\|\phi\|_C^2\big)$$

$\forall\,\phi\in C\cap L_V^2$.

(C3) (Coercitivity condition) There $\exists\,\beta > 0$, λ, C_1: $\forall\,\phi \in C \cap L_V^2$:

$$\langle A\phi(0), \phi(0)\rangle + \langle f(\phi), \phi(0)\rangle + \|\sigma(\phi)\|^2_{L_2^V} \le -\beta\|\phi(0)\|_V^2 + \lambda\|\phi\|_C^2 + C_1.$$

(C4) (Monotonicity condition) There $\exists\,\delta > 0$: $\forall\,\phi, \phi_1 \in C \cap L_V^2$:

$$2\big\langle A(\phi(0) - \phi_1(0)), \phi(0) - \phi_1(0)\big\rangle \\ + 2\big\langle f(\phi) - f(\phi_1), \phi(0) - \phi_1(0)\big\rangle + \|\sigma(\phi) - \sigma(\phi_1\|^2_{L_2^0} \le \delta\|\phi - \phi_1\|_C^2\,.$$

3 Main results

Definition. We call an F_t adapted random process $(u(t)) \in V$ weak solution for equation (0.1), (0.2) if:

(1) $u(t) = \phi(t)$, $t \in [-h, 0]$;

(2) $u \in L_2(\Omega \times [0,T], V)$;

(3) $\forall\, v \in V$, $t \in [0,T]$:

$$(u(t) - g(u_t), v) = (\phi - g(\phi), v) + \int_0^t (f(u_s) + Au, v)\,ds + \int_0^t (\sigma(u_s), v)\,dW(s).$$

Theorem (Existence and uniqueness)**.** *Suppose that conditions* (A1)–(A3)*,* (G1)–(G3) *and* (C1)–(C4) *hold, then* $\forall \phi \in C \cap L_V^2$ *equation* (0.1)*,* (0.2) *has a unique weak solution on* $[0,T]$ *such that*

$$u \in C([0,T] \times \Omega; H) \cap L_2([0,T] \times \Omega; V).$$

Moreover, the energy equation holds:

$$\|u - g(u_t)\|^2 = \|\phi - g(\phi)\|^2 \\ + \int_0^t \big\langle Au(s) + f(u_s), u(s)\big\rangle\,ds + \int_0^t \|\sigma(u_s)\|^2_{L_2^0}\,ds + \int_0^t \langle\sigma(u_s), u(s)\rangle\,dW(s).$$

Sketch of the proof:

Step 1: We consider projections of equation (0.1), (0.2) into sequence of finite-dimensional subspaces which looks as follows:

$$d\big(u^n(t) - g^n(u_t^n)\big) = \big(f^n(u_t^n) + Au^n(t)\big)\,dt + \sigma^n(u_t^n)\,dW^n(t), \tag{3.1}$$

$$u^n(t) = \phi^n(t), \quad t \in [-h, 0], \tag{3.2}$$

assuming that P_n is generated by $\{e_k; k = 1, \ldots, n\}$ of H and P_n' its restriction on V – projectors of H and V correspondingly:

(1) $A^n = P_n'A$;

(2) $u^n(t) = P_n'u(t)$;

(3) $\phi^n(t) = P_n'\phi(t)$;

(4) $f^n = P_n f$;

(5) $g^n = P_n g$;

(6) $\sigma^n = P_n \sigma$.

Then prove that each of $(3.1), (3.2)$ has exactly one solution.

Step 2: Then we create a priory estimate on solutions of projected equations, which looks as follows:

$$E \sup_{t \in [0,t_1]} \left(\|u^n(t)\|_V^2 + \|g^n u_t^n\|_V^2 \right) + E\bigg(\int_0^{t_1} \|u^n(t)\|_V^2 \, dt \bigg) \leq A$$

for some $A > 0$.

Those estimates are uniform (not dependant on dimension) and t_1 depends only on predefined coercitivity constants from (A3) and (C3), which implies that sequence of solutions are weak compact, hence holds weakly converging subsequence and can be iteratively continued on further intervals.

Step 3: After that we prove that we can make $n \to \infty$ in projected equations.

There we use the monotonicity condition and the growth conditions (G2) and (C2).

Additionally we prove that the energy equation holds, which implies existence and continuous dependence on initial data.

Corollary (Continuous dependence on the initial data)**.** *Let the conditions of the theorem above hold. Let ϕ and ϕ_1 be initial data for the solutions $u(t, \phi)$ and $u(t, \phi_1)$ of equation* (0.1)*,* (0.2)*. Then there exist a constant $C(T)$ such that*

$$E \sup_{t \in [0,T]} \left(\|u_t(\phi) - u_t(\phi_1)\|_C^2 \right) \leq C(T) \|\phi - \phi_1\|_C^2.$$

References

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