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Approximation of the classes $W^r_\beta H^\alpha$ by three-harmonic Poisson integrals

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(*Presented by V. P. Motornyi*)

Abstract. We obtain asymptotic equalities for the least upper bounds of deviations of the three-harmonic Poisson integrals from functions of the classes $W^r_\beta H^\alpha$ in a uniform metric in the case $r > 3,\ 0 \le \alpha < 1$.

Keywords. Kolmogorov–Nikol'skii problem, three-harmonic Poisson integral, asymptotic equality, Weyl–Nagy derivative, Lipschitz condition.

1. Introduction

Let L be a space of 2π-periodic functions f summable on the period with the norm $\|f\|_L = \int\limits_{-\pi}^{\pi} |f(t)|dt$, let C be a space of 2π-periodic continuous functions f in which the norm is set with the help of the equality $\|f\|_C = \max\limits_t |f(t)|$, and let L_∞ be a space of 2π-periodic measurable and essentially bounded functions f with the norm $\|f\|_\infty = \operatorname*{ess\,sup}\limits_t |f(t)|$.

Let $f \in L$, and let its Fourier series take the form

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) .$$

If $r > 0$, β is a fixed real number, and the series

$$\sum_{k=1}^{\infty} k^r \left(a_k \cos \left(kx + \frac{\beta\pi}{2} \right) + b_k \sin \left(kx + \frac{\beta\pi}{2} \right) \right) \tag{1.1}$$

is the Fourier series of some summable function φ, then the function φ is called the (r, β)-derivative of a function f in the Weyl–Nagy meaning and is denoted by f^r_β (see [1, p. 130]). The set of all functions satisfying such condition is denoted by W^r_β.

Let $f \in W^r_\beta$, and let $f^r_\beta \in H^\alpha$, i.e., f^r_β satisfies the Lipschitz condition of the order α:

$$|f^r_\beta(x+h) - f^r_\beta(x)| \le |h|^\alpha, \quad 0 < \alpha \le 1,\ 0 \le h \le 2\pi,\ x \in \mathbb{R}.$$

Then f belongs to the class $W^r_\beta H^\alpha$. For $\alpha = 0$, it is considered that $W^r_\beta H^0 = W^r_{\beta,\infty}$. For $r = \beta$, we get the class $W^r H^\alpha$ of functions f with a derivative of the order $r > 0$ in the Weyl meaning which satisfies the Lipschitz condition of the order α.

Let $f \in L$. The quantities

$$P_1(\delta; f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} (a_k \cos kx + b_k \sin kx) , \tag{1.2}$$

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$$P_2(\delta;f;x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\Big(1+\frac{k}{2}(1-e^{-\frac{2}{\delta}})\Big)e^{-\frac{k}{\delta}}\left(a_k\cos kx+b_k\sin kx\right), \tag{1.3}$$

$$P_3(\delta;f;x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\Big(1+\frac{1}{4}(3-e^{-\frac{2}{\delta}})(1-e^{-\frac{2}{\delta}})k$$

$$+\frac{1}{8}(1-e^{-\frac{2}{\delta}})^2k^2\Big)e^{-\frac{k}{\delta}}\big(a_k\cos kx+b_k\sin kx\big),\delta>0, \tag{1.4}$$

are called, respectively, the Poisson integral, biharmonic Poisson integral, and three-harmonic Poisson integral [2] of the function f.

Quantities (1.2)–(1.4) can be represented in the form of singular integrals

$$P_n(\delta;f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t+x)K_n(\delta;t)dt,\quad n=1,2,3,\ \ \delta>0,$$

with kernels

$$K_1(\delta;t)=\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\frac{k}{\delta}}\cos kt, \tag{1.5}$$

$$K_2(\delta;t)=\frac{1}{2}+\sum_{k=1}^{\infty}\Big(1+\frac{k}{2}(1-e^{-\frac{2}{\delta}})\Big)e^{-\frac{k}{\delta}}\cos kt, \tag{1.6}$$

$$K_3(\delta;t)=\frac{1}{2}+\sum_{k=1}^{\infty}\Big(1+\frac{k}{4}(3-e^{-\frac{2}{\delta}})(1-e^{-\frac{2}{\delta}})+\frac{k^2}{8}(1-e^{-\frac{2}{\delta}})^2\Big)e^{-\frac{k}{\delta}}\cos kt. \tag{1.7}$$

The present work is devoted to the study of the asymptotic behavior of the quantities

$$\mathcal{E}(W^r_\beta H^\alpha;P_3(\delta))_C=\sup_{f\in W^r_\beta H^\alpha}\|f(\cdot)-P_3(\delta;f;\cdot)\|_C, \tag{1.8}$$

as $\delta\to\infty$.

Let the function $\varphi(\delta)$ such that $\mathcal{E}(W^r_\beta H^\alpha;P_3(\delta))_C=\varphi(\delta)+o\left(\varphi(\delta)\right),\ \delta\to\infty$, be known in the explicit form. Following A. I. Stepanets [1, p. 198], we say that the Kolmogorov–Nikol'skii problem is solved for the class $W^r_\beta H^\alpha$ and three-harmonic Poisson integral in a uniform metric.

The Kolmogorov–Nikol'skii problem on various functional classes was solved within the methods of summation of Fourier series in the works by A. I. Stepanets and his disciples (see, e.g., [3–7]). We note that the case of triangular numerical matrices was considered in the majority of works. The asymptotic behavior of approximations of the classes of differentiable functions of a natural argument, which depend on a real parameter δ, as $\delta\to\infty$, was studied by the methods of summation to a somewhat less extent. In particular, the approximative properties of Poisson integrals and biharmonic Poisson integrals were studied on the classes of differentiable functions in works [8–21] and on the classes $W^r_\beta H^\alpha$ in works [22–24]. As for the approximative properties of three-harmonic Poisson integrals, they were considered in [25, 26].

As is known [27], while approximating the periodic differentiable functions with singular integrals with positive kernels [it is obvious that $K_1(\delta;t)>0$ and $K_2(\delta;t)>0$], it is impossible to attain a better approximation order than $\frac{1}{\delta^2},\delta\to\infty$. At the same time, the kernel $K_3(\delta;t)$ is alternating. Therefore, it is quite significant and actual to study the rate of approximation of the classes of differentiable functions (in particular, the classes $W^r_\beta H^\alpha$) with the help of three-harmonic Poisson integrals. In this case, it is possible to attain a higher approximation order than $\frac{1}{\delta^2}$, as $\delta\to\infty$. Therefore, we are faced with the problem of search for the asymptotic equalities for quantities (1.8).

2. Asymptotic equalities for the upper bounds of deviations of three-harmonic Poisson integrals from functions of the classes $W^r_\beta H^\alpha$.

The following proposition is true.

Theorem 2.1. *For $r > 3, 0 \le \alpha < 1,$ and $\delta \to \infty$, the asymptotic equality*

$$\mathcal{E}(W^r_\beta H^\alpha; P_3(\delta))_C = \frac{1}{\delta^3} \sup_{f \in W^r_\beta H^\alpha} \left\| \frac{4}{3} f_0^{(1)} + f_0^{(2)} + \frac{1}{6} f_0^{(3)} \right\|_C + O(\Upsilon(r)) \tag{2.1}$$

holds. Here, $f_0^{(r)}$, $r = 1, 2, 3,$ are the (r, β)-derivatives in the Weyl–Nagy meaning for $\beta = 0$, and

$$\Upsilon(r) = \begin{cases} \frac{1}{\delta^{r+\alpha}}, & 3 < r + \alpha < 4, \\ \frac{\ln \delta}{\delta^4}, & r + \alpha = 4, \\ \frac{1}{\delta^4}, & r + \alpha > 4. \end{cases} \tag{2.2}$$

Proof. For the three-harmonic Poisson integral $P_3(\delta)$ analogously to relation (6) in [28], we write the summing function $\tau(u)$ as

$$\tau(u) = \begin{cases} \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) \delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) u^{-r}, & u \ge \frac{1}{\delta}, \end{cases} \tag{2.3}$$

where $\gamma = \gamma(\delta) = \frac{1}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}})\delta, \quad \theta = \theta(\delta) = \frac{1}{8}(1 - e^{-\frac{2}{\delta}})^2 \delta^2, \ \delta > 0.$

The function $\tau(u)$ which is set with the help of relation (2.3) can be represented in the form $\tau(u) = \varphi(u) + \mu(u)$, where

$$\varphi(u) = \begin{cases} (\frac{4}{3\delta^2} u + \frac{1}{\delta} u^2 + \frac{1}{6} u^3)\delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(\frac{4}{3\delta^2} u + \frac{1}{\delta} u^2 + \frac{1}{6} u^3\right) u^{-r}, & u \ge \frac{1}{\delta}, \end{cases} \tag{2.4}$$

$$\mu(u) = \begin{cases} \left(1 - (1 + \gamma u + \theta u^2) e^{-u} - \frac{4}{3\delta^2} u - \frac{1}{\delta} u^2 - \frac{1}{6} u^3\right) \delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(1 - (1 + \gamma u + \theta u^2) e^{-u} - \frac{4}{3\delta^2} u - \frac{1}{\delta} u^2 - \frac{1}{6} u^3\right) u^{-r}, & u \ge \frac{1}{\delta}. \end{cases} \tag{2.5}$$

According to Theorem 3 in [22], if the Fourier transformations of functions $\varphi(u)$ and $\mu(u)$ of the form

$$\widehat{\varphi}(t) = \frac{1}{\pi} \int_0^\infty \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du, \tag{2.6}$$

$$\widehat{\mu}(t) = \frac{1}{\pi} \int_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \tag{2.7}$$

are summable on the whole number axis, the integrals

$$A(\alpha, \varphi) := \frac{1}{\pi} \int_{-\infty}^\infty |t|^\alpha \left| \int_0^\infty \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt, \tag{2.8}$$

$$A(\alpha, \mu) := \frac{1}{\pi} \int_{-\infty}^\infty |t|^\alpha \left| \int_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \tag{2.9}$$

are convergent, and $A(\alpha,\mu) = o\big(A(\alpha,\varphi)\big), \delta\to\infty$, then, for $0\le\alpha<1$ and $\delta\to\infty$, the asymptotic equality

$$\mathcal{E}(W^r_\beta H^\alpha; P_3(\delta))_C = \frac{1}{\delta^r}\sup_{f\in W^r_\beta H^\alpha}\|f_\varphi\|_C + O\left(\frac{1}{\delta^{r+\alpha}}A(\alpha,\mu)\right), \tag{2.10}$$

where

$$f_\varphi(x) := \int\limits_{-\infty}^{\infty}\Big(f^r_\beta\big(x+\frac{t}{\delta}\big) - f^r_\beta(x)\Big)\widehat{\varphi}(t)dt, \tag{2.11}$$

holds.

We now verify that the conditions of Theorem 3 in [22] are satisfied for functions $\varphi(u)$ and $\mu(u)$ of the forms (2.4) and (2.5), respectively.

The summability of transformations (2.6) and (2.7) is proved in work [29].

In order to show the convergence of the integral $A(\alpha,\varphi)$, we prove, according to Theorem 1 in [22, p. 6], the convergence of the integrals

$$\int\limits_0^{\frac12} u^{1-\alpha}|d\varphi'(u)|,\quad \int\limits_{\frac12}^{\frac32}|u-1|^{1-\alpha}|d\varphi'(u)|,\quad \int\limits_{\frac32}^{\infty}(u-1)|d\varphi'(u)|, \tag{2.12}$$

$$\int\limits_0^{\infty}\frac{|\varphi(u)|}{u^{1+\alpha}}du,\quad \int\limits_0^{1}\frac{|\varphi(1-u)-\varphi(1+u)|}{u^{1+\alpha}}du \tag{2.13}$$

and find their upper bounds.

We now estimate the first integral from (2.12). For $u\in\left[0;\frac{1}{\delta}\right]$, $\delta>2$, we have

$$\int\limits_0^{\frac1\delta} u^{1-\alpha}|d\varphi'(u)| = \delta^r\int\limits_0^{\frac1\delta}\left(\frac{2}{\delta}u^{1-\alpha}+u^{2-\alpha}\right)du \le \frac{K_1}{\delta^{3-(r+\alpha)}}. \tag{2.14}$$

For $u\in\left[\frac1\delta;\frac12\right]$, $\delta>2$, and $r>3$ with regard for $\varphi''(u)>0$, we get

$$\int\limits_{\frac1\delta}^{\frac12} u^{1-\alpha}|d\varphi'(u)| = \int\limits_{\frac1\delta}^{\frac12} u^{1-\alpha}d\varphi'(u) \le \frac{K_2}{\delta^{3-(r+\alpha)}}.$$

Hence,

$$\int\limits_0^{\frac12} u^{1-\alpha}|d\varphi'(u)| = O\left(\frac{1}{\delta^{3-(r+\alpha)}}\right),\ \delta\to\infty. \tag{2.15}$$

Let us estimate the second and third integrals in (2.12). Since

$$\int\limits_{\frac12}^{\frac32}|u-1|^{1-\alpha}|d\varphi'(u)| \le 2^\alpha\int\limits_{\frac12}^{\infty}u|d\varphi'(u)|,\quad \int\limits_{\frac32}^{\infty}(u-1)|d\varphi'(u)| \le \int\limits_{\frac12}^{\infty}u|d\varphi'(u)|,$$

in view of the obvious estimate

$$\int_{\frac{1}{2}}^{\infty} u|d\varphi'(u)| = O(1), \delta \to \infty,$$

we have

$$\int_{\frac{1}{2}}^{\frac{3}{2}} |u-1|^{1-\alpha}|d\varphi'(u)| = O(1), \quad \int_{\frac{3}{2}}^{\infty} (u-1)|d\varphi'(u)| = O(1), \delta \to \infty. \tag{2.16}$$

Using the scheme of estimation of the first integral in formula (12) of work [15], we can find the estimates of the integral $\int\limits_0^\infty \frac{|\varphi(u)|}{u^{1+\alpha}}du$ on each of the segments $\left[0; \frac{1}{\delta}\right]$, $\left[\frac{1}{\delta}; 1\right]$, and $[1; \infty)$. For $r > 3$ and $\delta \to \infty$, we obtain

$$\int_0^{\frac{1}{\delta}} \frac{|\varphi(u)|}{u^{1+\alpha}}du = \delta^r \int_0^{\frac{1}{\delta}} \frac{1}{u^{1+\alpha}} \left(\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3\right) du = O\left(\frac{1}{\delta^{3-(r+\alpha)}}\right), \tag{2.17}$$

$$\int_{\frac{1}{\delta}}^{1} \frac{|\varphi(u)|}{u^{1+\alpha}}du = \int_{\frac{1}{\delta}}^{1} \frac{1}{u^{1+\alpha}} \left(\frac{4}{3\delta^2}u^{1-r} + \frac{1}{\delta}u^{2-r} + \frac{1}{6}u^{3-r}\right) du = O\left(\frac{1}{\delta^{3-(r+\alpha)}}\right), \tag{2.18}$$

$$\int_1^{\infty} \frac{|\varphi(u)|}{u^{1+\alpha}}du = \int_1^{\infty} \frac{1}{u^{1+\alpha}} \left(\frac{4}{3\delta^2}u^{1-r} + \frac{1}{\delta}u^{2-r} + \frac{1}{6}u^{3-r}\right) du = O(1). \tag{2.19}$$

Relations (2.17)–(2.19) yield the estimate

$$\int_0^{\infty} \frac{|\varphi(u)|}{u^{1+\alpha}}du = O\left(\frac{1}{\delta^{3-(r+\alpha)}}\right), \quad r > 3, \delta \to \infty. \tag{2.20}$$

Analogously to formula (2.30) in [30], we can verify the validity of the equality

$$\int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u^{1+\alpha}}du = \int_0^1 \frac{|\lambda(1-u) - \lambda(1+u)|}{u^{1+\alpha}}du$$

$$+O\Big(|\varphi(0)| + |\varphi(1)| + \int_0^{\frac{1}{2}} u^{1-\alpha}|d\varphi'(u)| + \int_{\frac{1}{2}}^{\frac{3}{2}} |u-1|^{1-\alpha}|d\varphi'(u)| + \int_{\frac{3}{2}}^{\infty} (u-1)|d\varphi'(u)|\Big), \tag{2.21}$$

where $\lambda(u) = 1 - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3$. Since

$$\int_0^1 \frac{|\lambda(1-u) - \lambda(1+u)|}{u^{1+\alpha}}du = O(1), \delta \to \infty,$$

by virtue of relation (2.21) with regard for estimates (2.15) and (2.16), we get

$$\int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u^{1+\alpha}}du = O\left(\frac{1}{\delta^{3-(r+\alpha)}}\right), \quad r > 3, \ \delta \to \infty. \tag{2.22}$$

We have shown the convergence of integrals (2.12) and (2.13). Hence, according to Theorem 1 in [22], the integral $A(\alpha,\varphi)$ is convergent. For it, the following estimate holds:

$$A(\alpha,\varphi)=\Big(\frac{1}{\delta^{3-(r+\alpha)}}\Big),\quad \delta\to\infty. \tag{2.23}$$

We now prove the convergence of integral (2.9). To make this, according to Theorem 1 in [22, p. 6], we show the convergence of the integrals

$$\int\limits_0^{\frac12} u^{1-\alpha}|d\mu'(u)|,\quad \int\limits_{\frac12}^{\frac32} |u-1|^{1-\alpha}|d\mu'(u)|,\quad \int\limits_{\frac32}^{\infty}(u-1)|d\mu'(u)|, \tag{2.24}$$

$$\int\limits_0^{\infty}\frac{|\mu(u)|}{u^{1+\alpha}}du,\quad \int\limits_0^{1}\frac{|\mu(1-u)-\mu(1+u)|}{u^{1+\alpha}}du. \tag{2.25}$$

In order to estimate the integrals in (2.24), we will study firstly the function

$$\widetilde{\mu}(u)=1-(1+\gamma u+\theta u^2)e^{-u}-\frac{4}{3\delta^2}u-\frac{1}{\delta}u^2-\frac16 u^3. \tag{2.26}$$

Since

$$\widetilde{\mu}'(u)=(1+\gamma+\theta u^2)e^{-u}-(\gamma+2\theta u)e^{-u}-\frac{4}{3\delta^2}-\frac{2}{\delta}u-\frac12 u^2,$$

$$\widetilde{\mu}''(u)=-(1+\gamma+\theta u^2)e^{-u}-2(\gamma+2\theta u)e^{-u}-2\theta e^{-u}-\frac{2}{\delta}-u,$$

$$\widetilde{\mu}(0)=0,\ \widetilde{\mu}'(0)=1-\gamma-\frac{4}{3\delta^2}<0,$$

we can show that, for $u\geq 0$,

$$\widetilde{\mu}(u)\leq 0,\quad \widetilde{\mu}'(u)<0,\quad \widetilde{\mu}''(u)<0. \tag{2.27}$$

By virtue of (2.27) and the inequalities

$$e^{-u}\leq 1-u+\frac{u^2}{2}-\frac{u^3}{6}+\frac{u^4}{24},\quad e^{-u}\geq 1-u+\frac{u^2}{2}-\frac{u^3}{6},\quad e^{-u}\leq 1-u+\frac{u^2}{2},$$

$$e^{-u}\geq 1-u,\quad e^{-u}\leq 1,\quad u\geq 0,$$

we get

$$|\widetilde{\mu}(u)|\leq u\Big(\gamma-1+\frac{4}{3\delta^2}\Big)+u^2\Big(\frac12-\gamma+\theta+\frac1\delta\Big)+u^3\Big(\frac{\gamma}{2}-\theta\Big)+u^4\Big(\frac{1}{24}+\frac{\theta}{2}\Big),$$

$$|\widetilde{\mu}'(u)|\leq\Big(\gamma-1+\frac{4}{3\delta^2}\Big)+2u\Big(\frac12-\gamma+\theta+\frac1\delta\Big)+3u^2\Big(\frac{\gamma}{2}-\theta\Big)+u^3\Big(\frac16+2\theta\Big),$$

$$|\widetilde{\mu}''(u)|\leq 2\Big(\frac12-\gamma+\theta+\frac1\delta\Big)+6u\Big(\frac{\gamma}{2}-\theta\Big)+u^2\Big(\frac12+6\theta\Big).$$

Then, with regard for the estimates

$$\gamma-1+\frac{4}{3\delta^2}\leq\frac{3}{\delta^3},\quad \frac12-\gamma+\theta+\frac1\delta\leq\frac{3}{\delta^2},\quad \frac{\gamma}{2}-\theta\leq\frac{2}{\delta},$$

$$\frac{1}{24}+\frac{\theta}{2}\leq 1,\quad \frac12+6\theta\leq 3,\quad \frac16+2\theta\leq 2,$$

we have

$$|\widetilde{\mu}(u)| \leq \frac{3}{\delta^3}u + \frac{3}{\delta^2}u^2 + \frac{2}{\delta}u^3 + u^4, \tag{2.28}$$

$$\left|\widetilde{\mu}'(u)\right| \leq \frac{3}{\delta^3} + \frac{6}{\delta^2}u + \frac{6}{\delta}u^2 + 2u^3, \tag{2.29}$$

$$\left|\widetilde{\mu}''(u)\right| \leq \frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2. \tag{2.30}$$

In order to estimate the first integral in (2.24), we partition the segment $[0; \frac{1}{2}]$ into two parts: $[0; \frac{1}{\delta}]$ and $[\frac{1}{\delta}; \frac{1}{2}], \delta > 2$. From (2.5) in view of estimayes (2.28)–(2.30), we obtain

$$\int_0^{\frac{1}{\delta}} u^{1-\alpha}|d\mu'(u)| \leq \delta^r \int_0^{\frac{1}{\delta}} \Big(\frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2\Big)u^{1-\alpha}du \leq \frac{K_3}{\delta^{4-(r+\alpha)}}, \tag{2.31}$$

$$\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{1-\alpha}|d\mu'(u)| \leq r(r+1)\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{-r-\alpha-1}|\widetilde{\mu}(u)|du$$

$$+2r\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{-r-\alpha}|\widetilde{\mu}'(u)|du + \int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{1-r-\alpha}|\widetilde{\mu}''(u)|du$$

$$\leq r(r+1)\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{-r-\alpha-1}\Big(\frac{3}{\delta^3}u + \frac{3}{\delta^2}u^2 + \frac{2}{\delta}u^3 + u^4\Big)du$$

$$+2r\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{-r-\alpha}\Big(\frac{3}{\delta^3} + \frac{6}{\delta^2}u + \frac{6}{\delta}u^2 + 2u^3\Big)du + \int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{1-r-\alpha}\Big(\frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2\Big)du. \tag{2.32}$$

Relation (2.32) implies that

$$\int_{\frac{1}{\delta}}^{\frac{1}{2}} u^{1-\alpha}|d\mu'(u)| = \begin{cases} O(1), & 3 < r+\alpha < 4, \\ O\big(\ln\delta\big), & r+\alpha = 4, \\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha > 4, \end{cases} \quad \delta \to \infty. \tag{2.33}$$

Joining (2.31) and (2.33), we get the estimate

$$\int_0^{\frac{1}{2}} u^{1-\alpha}|d\mu'(u)| = \begin{cases} O(1), & 3 < r+\alpha < 4, \\ O\big(\ln\delta\big), & r+\alpha = 4, \\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha > 4, \end{cases} \quad \delta \to \infty. \tag{2.34}$$

Using (2.27) and the inequalities

$$e^{-u} \leq 1, \quad e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad e^{-u} \geq 1 - u, \quad u \geq 0,$$

we now estimate the function $\tilde{\mu}(u)$ and its derivatives in the following way:

$$|\widetilde{\mu}(u)| \le u\big(-1+\gamma+\frac{4}{3\delta^2}\big)+u^2\big(\frac{1}{2}-\gamma+\theta+\frac{1}{\delta}\big)+u^3\big(\frac{\gamma}{2}+\frac{1}{6}\big),$$

$$|\widetilde{\mu}'(u)| \le \big(-1+\gamma+\frac{4}{3\delta^2}\big)+u\big(1-2\gamma+2\theta+\frac{2}{\delta}\big)+u^2\big(\frac{3}{2}\gamma+\theta+\frac{1}{2}\big),$$

$$|\widetilde{\mu}''(u)| \le \big(1-2\gamma+2\theta+\frac{2}{\delta}\big)+u\big(3\gamma+1\big)+\big(\theta u^2+4\theta u\big)e^{-u}.$$

Then, using the estimates

$$-1+\gamma+\frac{4}{3\delta^2} \le \frac{2}{\delta^2},\ \ \frac{1}{2}-\gamma+\theta+\frac{1}{\delta} \le \frac{2}{\delta},\ \ \frac{\gamma}{2}+\frac{1}{6} \le 1,\ \frac{3}{2}\gamma+\theta+\frac{1}{2} \le 4,$$

$$3\gamma+1 \le 6,\ \big(4\theta u+\theta u^2\big)e^{-u} \le 2u,\ u \ge 0,$$

we have

$$|\widetilde{\mu}(u)| \le \frac{2}{\delta^2}u+\frac{2}{\delta}u^2+u^3,\ u \ge 0, \tag{2.35}$$

$$|\widetilde{\mu}'(u)| \le \frac{2}{\delta^2}+\frac{4}{\delta}u+4u^2,\ u \ge 0, \tag{2.36}$$

$$|\widetilde{\mu}''(u)| \le \frac{4}{\delta}+8u,\ u \ge 0. \tag{2.37}$$

Since

$$\int\limits_{\frac{1}{2}}^{\frac{3}{2}} |u-1|^{1-\alpha}|d\mu'(u)| \le 2^{\alpha}\int\limits_{\frac{1}{2}}^{\infty} u|d\mu'(u)|,\ \int\limits_{\frac{3}{2}}^{\infty}(u-1)|d\mu'(u)| \le \int\limits_{\frac{1}{2}}^{\infty} u|d\mu'(u)|,$$

with regard for relations (2.35)–(2.37), we get the estimate of the integral

$$\int\limits_{\frac{1}{2}}^{\infty} u|d\mu'(u)| \le r(r+1)\int\limits_{\frac{1}{2}}^{\infty} u^{-r-1}\Big(\frac{2}{\delta^2}u+\frac{2}{\delta}u^2+u^3\Big)du$$

$$+2r\int\limits_{\frac{1}{2}}^{\infty} u^{-r}\Big(\frac{2}{\delta^2}+\frac{4}{\delta}u+4u^2\Big)du+\int\limits_{\frac{1}{2}}^{\infty} u^{1-r}\Big(\frac{4}{\delta}+8u\Big)du \le K_6,\quad r>3. \tag{2.38}$$

Thus,

$$\int\limits_{\frac{1}{2}}^{\frac{3}{2}} |u-1|^{1-\alpha}|d\mu'(u)| = O(1),\quad \int\limits_{\frac{3}{2}}^{\infty}(u-1)|d\mu'(u)| = O(1),\ \delta\to\infty. \tag{2.39}$$

We now estimate the first integral in (2.25), by dividing the segment $[0;\infty)$ into three parts: $[0;\frac{1}{\delta}]$, $[\frac{1}{\delta};1]$, and $[1;\infty)$. From formula (2.26) with regard for relations (2.28) and (2.35), we get

$$\int\limits_{0}^{\frac{1}{\delta}} \frac{|\mu(u)|}{u^{1+\alpha}}du = \delta^r\int\limits_{0}^{\frac{1}{\delta}} |\tilde{\mu}(u)|\frac{du}{u^{1+\alpha}} \le \delta^r\int\limits_{0}^{\frac{1}{\delta}}\left(\frac{3}{\delta^3}u+\frac{3}{\delta^2}u^2+\frac{2}{\delta}u^3+u^4\right)\frac{du}{u^{1+\alpha}}du \le \frac{K_7}{\delta^{4-(r+\alpha)}},$$

$$\int\limits_{\frac{1}{\delta}}^{1}\frac{|\mu(u)|}{u^{1+\alpha}}du=\int\limits_{\frac{1}{\delta}}^{1}|\tilde{\mu}(u)|\frac{du}{u^{1+\alpha+r}}=\begin{cases}O(1), & 3<r+\alpha<4,\\ O\big(\ln\delta\big), & r+\alpha=4,\\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha>4,\end{cases}\quad \delta\to\infty,$$

$$\int\limits_{1}^{\infty}\frac{|\mu(u)|}{u^{1+\alpha}}du=\int\limits_{1}^{\infty}|\tilde{\mu}(u)|\frac{du}{u^{1+\alpha+r}}\le\int\limits_{1}^{\infty}\left(\frac{2}{\delta^2}u+\frac{2}{\delta}u^2+u^3\right)u^{-r-1-\alpha}du\le K_8.$$

Joining the last relations, we get

$$\int\limits_{0}^{\infty}\frac{|\mu(u)|}{u^{1+\alpha}}du=\begin{cases}O(1), & 3<r+\alpha<4,\\ O\big(\ln\delta\big), & r+\alpha=4,\\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha>4,\end{cases}\quad \delta\to\infty. \tag{2.40}$$

In order to estimate the second integral in (2.25), we note that the equality

$$\int\limits_{0}^{1}\frac{|\mu(1-u)-\mu(1+u)|}{u^{1+\alpha}}du=\int\limits_{0}^{1}\frac{|\overline{\lambda}(1-u)-\overline{\lambda}(1+u)|}{u^{1+\alpha}}du$$

$$+O\Big(|\mu(0)|+|\mu(1)|+\int\limits_{0}^{\frac{1}{2}}u^{1-\alpha}|d\mu'(u)|+\int\limits_{\frac{1}{2}}^{\frac{3}{2}}|u-1|^{1-\alpha}|d\mu'(u)|+\int\limits_{\frac{3}{2}}^{\infty}(u-1)|d\mu'(u)|\Big),$$

where $\overline{\lambda}(u)=(1+\gamma u+\theta u^2)e^{-u}+\frac{4}{3\delta^2}u+\frac{1}{\delta}u^2+\frac{1}{6}u^3$, holds.

Since

$$\int\limits_{0}^{1}\frac{|\overline{\lambda}(1-u)-\overline{\lambda}(1+u)|}{u^{1+\alpha}}du=O(1),\ \ \delta\to\infty,$$

with regard for relations (2.34) and (2.39), we obtain

$$\int\limits_{0}^{1}\frac{|\mu(1-u)-\mu(1+u)|}{u^{1+\alpha}}du=\begin{cases}O(1), & 3<r+\alpha<4,\\ O\big(\ln\delta\big), & r+\alpha=4,\\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha>4,\end{cases}\quad \delta\to\infty. \tag{2.41}$$

Using formulas (2.34), (2.39), (2.40), and (2.41) and Theorem 1 in [22], we verify that the integral $A(\alpha,\mu)$ is convergent and satisfies the estimate

$$A(\alpha,\mu)=\begin{cases}O(1), & 3<r+\alpha<4,\\ O\big(\ln\delta\big), & r+\alpha=4,\\ O\Big(\frac{1}{\delta^{4-(r+\alpha)}}\Big), & r+\alpha>4,\end{cases}\quad \delta\to\infty. \tag{2.42}$$

Hence, the conditions of Theorem 3 in [22] are satisfied, i.e., equality (2.10) holds. With regard for estimate (2.42), as $\delta\to\infty$, we get

$$\mathcal{E}(W^r_\beta H^\alpha;P_3(\delta))_C=\frac{1}{\delta^r}\sup_{f\in W^r_\beta H^\alpha}\|f_\varphi\|_C+O\left(\Upsilon(r)\right), \tag{2.43}$$

where $\Upsilon(r)$ and the function $f_\varphi(x)$ are defined by formulas (2.2) and (2.11), respectively.

It is possible to show that the Fourier series of the function $f_\varphi(x)$ has the form (see, e.g., [28])

$$S[f_\varphi(x)] = \sum_{k=1}^{\infty} \varphi\left(\frac{k}{\delta}\right) k^r (a_k \cos kx + b_k \sin kx),$$

where a_k, b_k are the Fourier coefficients of the function f. From whence, in view of formula (2.4), we get

$$S[f_\varphi(x)] = \frac{1}{\delta^{3-r}} \sum_{k=1}^{\infty} \left(\frac{4}{3}k + k^2 + \frac{1}{6}k^3\right) (a_k \cos kx + b_k \sin kx).$$

Therefore, according to (1.1), we get

$$f_\varphi(x) = \frac{1}{\delta^{3-r}} \left(\frac{4}{3} f_0^{(1)}(x) + f_0^{(2)}(x) + \frac{1}{6} f_0^{(3)}(x)\right). \tag{2.44}$$

Substituting (2.44) in (2.43), we obtain (2.1). The theorem is proved. □

Remark 2.1. Comparing the results of works [22, 24] and Theorem 2.1, we see that the orders of the approximations of the classes $W^r_\beta H^\alpha, r > 3$, with the help of a Poisson integral, biharmonic Poisson integral, and three-harmonic Poisson integral are equal to $\frac{1}{\delta}$, $\frac{1}{\delta^2}$, and $\frac{1}{\delta^3}$, respectively.

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